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Axial Symmetry and Transverse Trace-Free Tensors in Numerical Relativity

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Doctor of Philosophy

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Contents

I, Rory Patrick Albert Conboye, certify that this thesis is my own work and I have not obtained a degree in this university or elsewhere on the basis of the work submitted in this thesis.

Rory Patrick Albert Conboye

Abstract

Transverse trace-free (TT) tensors play an important role in the initial conditions of numerical relativity, containing two of the component freedoms. Expressing a TT tensor entirely, by the choice of two scalar potentials, is not a trivial task however.

Assuming the added condition of axial symmetry, expressions are given in both spherical and cylindrical coordinates, for TT tensors in flat space. A coordinate relation is then calculated between the scalar potentials of each coordinate system. This is extended to a non-flat space, though only one potential is found. The remaining equations are reduced to form a second order partial differential equation in two of the tensor components.

With the axially symmetric flat space tensors, the choice of potentials giving Bowen-York conformal curvatures, are derived. A restriction is found for the potentials which ensure an axially symmetric TT tensor, which is regular at the origin, and conditions on the potentials, which give an axially symmetric TT tensor with a *spherically* symmetric scalar product, are also derived.

A comparison is made of the extrinsic curvatures of the exact Kerr solution and numerical Bowen-York solution for axially symmetric black hole space-times. The Brill wave, believed to act as the difference between the Kerr and Bowen-York space-times, is also studied, with an approximate numerical solution found for a *mass-factor*, under different amplitudes of the metric.

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Introduction

This thesis is broken into five main chapters. The first two provide background material, with the remaining three covering the main work of the thesis. These are followed by a conclusions chapter, appendices providing specific calculations for chapter 4 and *Mathematica* input for chapter 5, and a list of the symbols used in the thesis, with descriptions and, where necessary, references to pages and equations.

Chapter 1

Section 1.1 begins with a brief introduction to general relativity, and the principles used by Einstein to develop the theory. The conventions for the thesis, in units and notation, are then laid out in subsection 1.1.2, mainly for reference.

The mathematics used in the thesis is outlined in section 1.2, beginning with the definitions of manifolds, vectors and tensors, and the notation which will be used for most of the calculations in the thesis. The metric, which plays the pivotal part of measuring distance on a manifold, is then defined in subsection 1.2.2.

Since infinitesimal motions are so important to physical theories, subsection 1.2.3 derives the two main concepts of derivatives on a manifold, the covariant and Lie derivatives, along with their applications to vectors and tensors. The equations used for transforming between different coordinate systems are then given in subsection 1.2.4, along with the intuitive and differential definitions of the intrinsic curvatures of a manifold.

Subsection 1.2.5 gives a definition for the metric preserving Killing vectors, and states the Frobenius theorem in both its vector and dual forms, both of which are used at later stages.

The third section of the chapter covers the basics of the theory of relativity. Subsection 1.3.1 gives a brief outline of the special theory of relativity, leading in particular to the Minkowski metric. Subsection 1.3.2 then begins with an arbitrary space-time metric, and its implications on causality, which is followed by the definition of the stress-energy tensor, and a brief derivation of the Einstein field equations.

Subsections 1.3.3 and 1.3.4 outline two of the known exact solutions of Einstein's equations, the Schwarzschild and Kerr solutions. The metric for each solution is given, in appropriate coordinate systems, with a short discussion of the properties of the space-times described by each.

Chapter 2

The second chapter develops the $3+1$ formalism for numerical relativity, beginning in subsection 2.1.1, with the foliation of the space-time manifold into 3 space dimensions and 1 time. By use of the Frobenius theorem in subsection 2.1.2, coordinates are shown to be adaptable to the $3+1$ splitting, allowing further calculations to distinguish between space-time objects and purely spatial objects.

The extrinsic curvature of a spatial surface, embedded in a space-time manifold, is then defined in subsection 2.1.3, and shown to give a time-like derivative of the spatial metric. The constraint and evolution equations, given by decomposing the Einstein field equations, are then derived in subsections 2.1.4 and 2.1.5. The algorithm for implementing the $3+1$ formalism is outlined in 2.1.6, and the exact solutions of Schwarzschild and Kerr shown to be $3+1$ decomposable in subsection 2.1.7.

Section 2.2 deals with the initial conditions for the $3+1$ formalism, with subsection 2.2.1 introducing the conformal transformation of the spatial metric, and subsection 2.2.2 deriving the transverse and trace-free decompositions of the extrinsic curvature. It is these decompositions of the curvature, which the content of chapter 3 is based on.

Two methods for solving the $3+1$ constraint equations, to find initial conditions, are then outlined, with the conformal transverse trace-free decomposition in subsection 2.2.3, and the conformal thin sandwich decomposition in subsection 2.2.4. The first of these is then shown to give the *Bowen-York* initial conditions in subsection 2.2.5.

With the initial conditions determined, a brief outline is given of some of the techniques for choosing the gauge conditions, for the evolution of the $3+1$ system, in section 2.3. The highly successful BSSN reformulation of the $3+1$ formalism, is briefly shown in section 2.4.

Chapter 3

Chapter 3 concerns the finding of expressions for transverse trace-free tensors, which are also axially symmetric. The chapter begins with a brief review of some of the work already done on these type of tensors in section 3.1, going into detail on the symmetry of "time rotation" solution, provided by *Sergio Dain* [19].

Section 3.2 derives an expression for a TT tensor in flat space, in both spherical and cylindrical coordinates, giving the tensors in terms of two scalar potentials. An attempt is made to compare these representations in subsection 3.2.3.

These techniques are then extended in section 3.3, to a non-flat 3-space, with a *Brill* wave metric given by *Brill* [13]. Though not as successful as the flat space tensors, a "time rotation" symmetry part is found to be similar to *Dain* [19], with the remaining equations reduced as far as possible in subsection 3.3.3.

In section 3.4, a particular choice of the scalar potentials, for the flat space tensors in section 3.2, are shown to give the Bowen-York curvature of subsection 2.2.5. Conditions are then derived in section 3.5, giving tensors which are regular at the origin, and in section 3.6, for tensors which give a spherically symmetric scalar product.

Chapter 4

The axially symmetric space-times of Kerr and Bowen-York are compared in chapter 4, starting with a review of some of the more influential works relating the two. The curvature tensors for each are given in section 4.2, and a few basic calculations carried out in section 4.3.

Chapter 5

The effective difference between the Kerr and Bowen-York space-times is seen in chapter 4 to be given by Brill wave gravitational radiation. Subsection 5.1.1 gives an outline from *Brill* [13] of the spatial metric and positivity of mass for a Brill wave, with a Schrödinger analogue investigated. Two different specifications for the Brill metric, and their numerically evaluated conformal factors, are then reviewed in subsection 5.1.2.

In section 5.2 an attempt is made to reduce the complexity of finding the conformal factor for a Brill metric, with a "mass-factor" derived for this simplified case. The results of numerical calculations performed in *Mathematica* are then given in section 5.2.3, coresponding highly with the Schrödinger analogue of *Brill* [13]. The *Mathematica* operations used for the numerical approximations are given in appendix B.

Appendix A

In appendix A, calculations are carried out for the extrinsic curvature of the Kerr metric, for use in chapter 4. The appropriate metric components and their derivatives are calculated in section A.1, leading to the components of the extrinsic curvature. Section A.2 then gives the coordinate derivatives of the curvature components, with section A.3 showing the calculated expressions to satisfy both the Hamiltonian and momentum constraints.

Chapter 1

General Relativity and Differential Geometry

1.1 Relativity Theory

1.1.1 From Special to General Relativity

Albert Einstein postulated the theory of general relativity in 1915, viewing gravity as a distortion of the structure of space and time, rather than a force, as Newton had described it. In Einstein's theory, mass changes the shape of space-time, and this change in shape effects how objects of mass move through space-time.

The problem with Newton's gravitational theory, was that the force propagates at an infinite speed. Any change in force at one point, would be experienced everywhere, immediately. This is known as instant "action at a distance", and contradicted Einstein's *special* theory of relativity, which states that no physical interaction can be transmitted faster than the speed of light.

Einstein spent the next ten years, from 1905 to 1915 trying to find a relativistic theory of gravity. There were three principles in particular which directed Einstein:

- the principle of equivalence,
- the principle of general covariance,
- Mach's principle.

See Einstein's own non-specialist work, *Einstein* [23], for an intuitive explanation.

The "principle of equivalence" originated from Galileo's experiments, showing that all objects, regardless of their mass, experience the same acceleration in a gravitational field. This was then extended by Einstein, to the equivalence, for an observer, of a gravitational field to an acceleration. In this sense, free-fall in a gravitational field would have the equivalent effect on an observer, as zero acceleration in a region with no gravitational field. This equivalence leads to certain special relativistic time and space measurements not "fitting" into the Euclidean geometry, on which both Newtonian mechanics and special relativity are based. Thus the *general* theory of relativity would have to make use of a *non*-Euclidean space-time geometry.

The "principle of general covariance" is the natural generalisation of the principle of special relativity, made possible by use of the principle of equivalence described above. It is an assumption, that the laws of physics take the same form for *every* observer, regardless of position and of motion, and in particular, regardless of non-uniform motion. This is equivalent to assuming that the laws of physics are completely invariant to the coordinate choices on a non-Euclidean space-time geometry. This lead Einstein to formulate his gravitational theory in a "generally covariant" tensor form, completely free from choices of coordinates, and hence, position and motion.

Finally, “Mach’s principle” from Ernst Mach, relates the inertial properties of all physical objects, to the total distribution of matter in the universe. As such, it was used by Einstein to relate the gravitational influence of matter, to the geometry of the general relativistic space-time.

1.1.2 Notation Conventions

In the relativity literature, there are a number of different conventions with regard to notation and to signs of particular entities. The conventions used in this thesis are thus outlined briefly here, as well as when they first arise in the rest of the text.

The units are defined, such that both the gravitational constant G and the speed of light c are given by unity, $G = c = 1$. These are known as “geometrized units”.

All tensors are taken to have *real* components, leading to the requirement for a non-positive-definite metric. As such, the metric is taken to have “Lorentz” signature $(- + + +)$, similar to *Misner, Thorne & Wheeler* [40], *Hawking & Ellis* [31] and *Wald* [49].

The covariant derivative is assumed to be torsion-free, which according to *Misner, Thorne & Wheeler* [40], is necessary for the validity of the principle of equivalence. There are, however, other formalisms of relativity which include torsion, e.g. Élie Cartan’s theory, see *Cartan* [15] & [16].

The sign convention for the extrinsic curvature is that of *Wald* [49], with the curvature given by the *positive* Lie derivative of the spatial metric. Equivalently, the sign of the *trace* of the curvature is defined to represent the *positive* rate of change of the volume element, along the future pointing time-like normal vector.

The abstract index notation of *Wald* [49] is also used, once it has been defined. The dimension of the indices, however, are taken similar to *Misner, Thorne & Wheeler* [40], with Greek-letter indices representing 4-dimensional objects, from section 1.3.1 onwards, and Latin-letter indices representing 3-dimensional, spatial objects, from section 2.1.2 onwards:

$$\alpha, \beta, ..., \mu, \nu, ... \in \{0, 1, 2, 3\} \ ,$$
$$a, b, ..., i, j, ... \in \{1, 2, 3\} \ .$$

This is similar to the notation of *Alcubierre* [1]. In exception to this, however, Latin letters are used to represent arbitrary dimensional indices for the development of the mathematical concepts in section 1.2. Thus, objects defined in section 1.2 can be used later, in whichever dimension is necessary.

1.2 Background Mathematics

In order to work with Einstein's general relativity, a number of mathematical concepts need to be defined. The content of this section is generally based on *Kobayashi & Nomizu* [35], [36] and *Griffiths & Harris* [29], with a more intuitive approach given in *Olver* [43] and reference to notation in particular from *Wald* [49] and *Alcubierre* [1].

1.2.1 Manifolds, Vectors and Tensors

Euclidean Space

Euclidean geometry refers to the geometry developed by the ancient Greeks, collected and formalised in 300 BC by Euclid of Alexandria, in his *Elements* [25]. As such, a space in which all of Euclid's "postulates" hold, is known as a *Euclidean* space. The geometry of Euclid was then formulated in a *coordinate* form by René Descartes, in the 17^{th} century [22], defining each point of space by a unique assignment of 3 numbers.

With the requirements for modern mathematics and physics, the concepts of Cartesian 3-dimensional space, are generalised to higher dimensions by a real "n-space", defined to be the space given by ordered collections of n real numbers:

$$(a_1, \dots, a_n) \qquad \text{such that} \qquad a_i \in \mathbb{R}, \quad \forall\ i \in \{1, \dots, n\}\ . \tag{1.2.1}$$

An "n-space" is then known as a *vector space*, if it contains the usual vector addition and scalar multiplication of its elements:

$$\begin{aligned} a + b &= (a_1 + b_1, \dots, a_n + b_n)\ , \\ ka &= (ka_1, \dots, ka_n)\ , \qquad\qquad \text{s.t.} \qquad k \in \mathbb{R}\ , \end{aligned} \tag{1.2.2}$$

with the elements a and b now known as *vectors*.

An n-dimensional *Euclidean* space $\mathbb{E}^n$, in Cartesian coordinates, is now defined to be a real vector space with a "distance" measure, given by Pythagoras' theorem. The distance d between two points a and b is thus given by:

$$d^2 = (b_1 - a_1)^2 + \ \dots\ + (b_n - a_n)^2\ , \tag{1.2.3}$$

with the usual real number product used for the squaring. The distance measure also gives a concept of a vector product, known as the *inner* or *dot* product:

$$a \cdot b = a_1 b_1 + \ \dots\ + a_n b_n\ , \tag{1.2.4}$$

again using the usual real number product, for each part of the sum.

Differential Manifolds

A *non*-Euclidean space is generally taken to be a space in which Euclid's *fifth* postulate does not hold. This postulate can be seen to be equivalent to the "triangle" postulate, which states that "The angles of all triangles sum to two right angles." The concept of a Euclidean space is generalised to that of a *manifold*, which is seen to be "locally" equivalent to a Euclidean space, with the triangle postulate not necessarily holding "globally".

In the precise definition, a manifold is considered to be an n-dimensional set, with a countable collection of subsets, which each have a 1-to-1 correspondence with a subset of n-dimensional Euclidean space. The subsets of the manifold are known as coordinate charts, and the relations to Euclidean space as local coordinate maps.

A manifold must also have a Hausdorff topology, with any two points on the manifold capable of being surrounded by separate, non-intersecting neighbourhoods. This means that the "separation of points" doesn't change with smaller and smaller scales.

A manifold is known as *differential*, if for any two coordinate charts with an overlap on the manifold, the relation in Euclidean space, between the coordinate maps of the overlap, is differentiable.

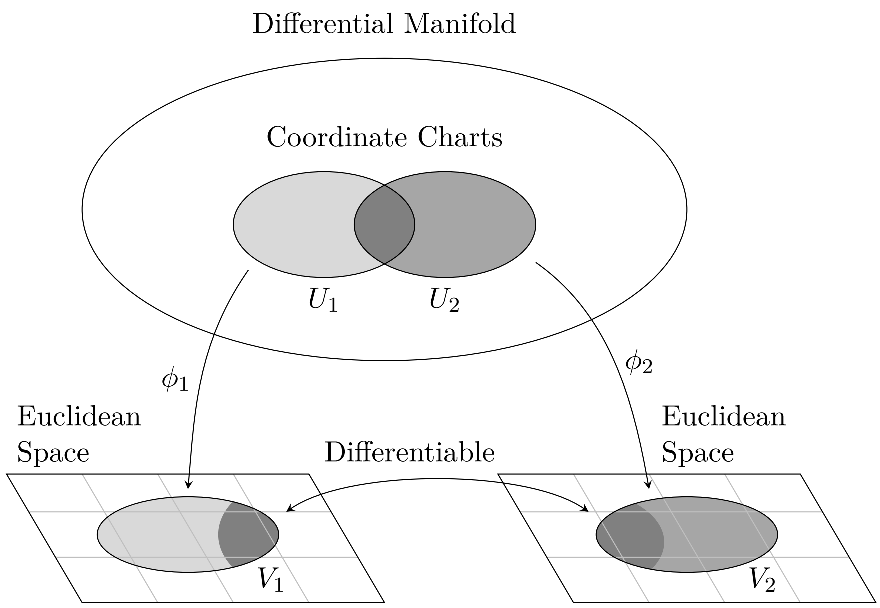


Figure 1.1: Differential manifold with coordinate maps into Euclidean space.

For the study of general relativity, space and time are considered to form a single 4-dimensional differential manifold. An element of the space-time manifold represents an "event" in space and time. In a particular coordinate system, this event would be represented by the usual one time and three spatial coordinates.

Differential Functions and Curves

A real-valued function on a differential manifold M, assigns an element of the real numbers to each point of the manifold:

$$f: \ M \ \rightarrow \ \mathbb{R} \ . \tag{1.2.5}$$

A *differential* function is a real-valued function, which is differentiable in the coordinate maps, in Euclidean space, for each coordinate chart of M.

A curve on a manifold is, intuitively, a continuous pathway of events. Mathematically, this is defined to be a map from an interval of the real numbers to a 1-dimensional connected subspace of the manifold:

$$\gamma: \ I \ \rightarrow \ M, \qquad \text{s.t.} \qquad \gamma(t) \ \in \ M, \qquad \forall \, t \in I \subset \mathbb{R} \ . \tag{1.2.6}$$

Taking a coordinate chart U surrounding a portion of the curve, with a coordinate map $\phi : U \rightarrow \mathbb{E}^n$, the coordinates of the curve points in $\mathbb{E}^n$ are given by n functions γ^i:

$$\phi(\gamma) \ = \ (\gamma^1, ..., \gamma^n) \ , \qquad \text{with} \quad \gamma^i : \ I \ \rightarrow \ \mathbb{R} \ , \quad \forall \, i \in \{1, ..., n\} \ . \tag{1.2.7}$$

A curve is *differential*, with no "sharp turns", if it is differentiable with respect to its parameter, in the Euclidean space of each coordinate chart it passes through.

Tangent Vectors and Vector Fields

A tangent vector at a point on a manifold, is an element of a vector space defined at that point. In a Euclidean space $\mathbb{E}^n$, the tangent space is considered as a subspace of the Euclidean space itself. However, since a differential manifold is not a vector space, the concept of a tangent vector has to be defined in a slightly different way.

A tangent vector is therefore defined using *equivalence classes* of differential curves. The equivalence classes are formed by dividing the set of all differential curves passing through a particular point, into different "classes" based on their derivatives in Euclidean space at that point. For example, taking a coordinate chart U surrounding a point x, with a coordinate map $\phi : U \rightarrow \mathbb{E}^n$:

$$\gamma_1(t) \ \equiv \ \gamma_2(s) \qquad \Leftrightarrow \qquad \frac{d}{dt}\phi\left(\gamma_1\right) \ = \ \frac{d}{ds}\phi\left(\gamma_2\right) \ , \tag{1.2.8}$$

with the derivatives evaluated at the point $\gamma_1(t) = \gamma_2(s) = x$. Hence, at the point x, the differential curves $\gamma_1(t)$ and $\gamma_2(s)$ are elements of the same equivalence class. Each tangent vector at a point, is then defined by the class of curves given by a particular derivative in Euclidean space.

When a particular coordinate system is chosen, each vector can be expressed as a linear combination of the coordinate basis vectors $\vec{e}_i$, with appropriate coefficients:

$$\vec{v} \;=\; \sum_{i=1}^{n} v^i \vec{e}_i \;=\; v^i \vec{e}_i \;, \tag{1.2.9}$$

where Einstein notation is used in the last part, summing over indices which appear both above and below.

A vector *field* is defined to be a smooth assignment of a vector at each point of an open subset of the manifold. Equation (1.2.9) holds for a vector field $\vec{v}$, with $\vec{e}_i$ representing basis vector *fields*. Throughout this thesis, all vectors are considered to be "point elements" of vector fields, and hence a field is implied everywhere a vector is given.

An *integral* curve of a vector field, is defined to be a differential curve which is an element of the equivalence class of the vector field, at each point on the curve. By the definition of both vectors, and vector fields, a family of integral curves *must* exist, for every vector field, in its domain of existence.

An integral *manifold* is a differential submanifold, such that the integral curves of its tangent vector fields, are completely contained in the submanifold itself. Unlike integral curves, however, integral manifolds do not necessarily exist for every vector field.

In the *abstract* index notation, see *Wald* [49], vectors (i.e. vector fields), are represented with a raised index $\vec{v} := v^a$, regardless of any coordinate basis being chosen. This is an extension of the notation used for the *coefficients* of the vector in equation (1.2.9), and proves to be an efficient notation for tensor calculations on a manifold.

Covectors and Covector Fields

Linear functionals act on vectors to give a real number, and are known as 1-forms or covectors:

$$\vec{\zeta} : \; \vec{v} \;\rightarrow\; \mathbb{R} \;. \tag{1.2.10}$$

Covectors also form a vector space, known as the dual tangent space, and can be written as the linear combination of basis *covectors* $\vec{\omega}^i$, with appropriate coefficients:

$$\vec{\zeta} \;= \zeta_i \, \vec{\omega}^i \;, \tag{1.2.11}$$

once a particular coordinate system has been defined.

The basis covectors are then defined, such that when acting on the basis *vectors* at the same point, the Kronecker delta is returned, simplifying the covector action:

$$\vec{\omega}^i(\vec{e}_j) \;=\; \delta^i_j \qquad\Rightarrow\qquad \zeta(v) \;=\; \zeta_i\, \vec{\omega}^i(v^j \vec{e}_j) \;=\; \zeta_i\, v^j\, \delta^i_j \;=\; \zeta_i\, v^i \;, \tag{1.2.12}$$

given as the sum of the product of the covector and vector components, otherwise known as a contraction.

The above equation shows a symmetry between vectors and covectors, culminating in the fact that vectors can also be seen as linear functionals of covectors. Due to this symmetry, and equation (1.2.11), covectors are represented in the abstract index notation with a *lowered* index $\vec{\zeta} := \zeta_a$, in keeping with the Einstein summation convention.

Again, covectors form covector *fields*, by a smooth assignment of a covector at each point of an open subset of a manifold. The action of a covector field on a vector field then gives a real value at each point, or a differential function on an open set:

$$\vec{\zeta} :\; \vec{v} \;\rightarrow\; f \;. \tag{1.2.13}$$

As with vectors, for the remainder of this thesis, a covector *field* is assumed anywhere a covector is given.

Tensors and Tensor Fields

Tensors are multi-linear functionals, acting on either vectors, covectors, or both. Again, all tensors are assumed to be tensor *fields* on open subsets of a manifold, with actions giving real numbers at each point, or differential functions on open subsets.

In the abstract index notation a tensor is given a lowered index for every vector it acts on, and a raised index for every covector it acts on. For example:

$$T_{ab}\;(v^a, w^b), \qquad T^{b}_{a}\;(v^a, \zeta_b)\;, \qquad T^{ab}\;(\zeta_a, \xi_b)\;, \tag{1.2.14}$$

where the tensors T act on vectors v^a, w^b and covectors ζ_a, ξ_b. The vectors and covectors on which the tensor acts are linked by the indices, with the brackets usually omitted as a result. The products can then be expanded in a coordinate system, by use of the Einstein summation convention.

Tensors are organized in types according to how many vectors and covectors they operate on or, more directly, how many upper and lower indices they have. A tensor operating on p vectors and q covectors is known as a $\binom{q}{p}$ type tensor.

1.2.2 The Metric Tensor

Since a differential manifold is not itself a vector space, the Euclidean distance measure given by equation (1.2.3), cannot be used directly. However, the *tangent* vector space at each point, can be used to give an "infinitesimal" measure of distance, since a manifold is considered "locally" equivalent to Euclidean space. It remains then, to give the inner product for each tangent space, in order to find the norm, or "square", of the tangent vectors.

Tangent Space Inner Product

An inner product acts on two vectors, regardless of order, to give a real number. Its action can therefore be given by a non-degenerate, symmetric $\binom{0}{2}$ type tensor:

$$\langle \vec{v}, \vec{w} \rangle \;=\; g(\vec{v}, \vec{w}) \;=\; g_{ab}\; v^a w^b \qquad \in\; \mathbb{R}\;, \tag{1.2.15}$$

with $\vec{v}$ and $\vec{w}$ elements of the tangent space of a particular point. The tensor g_{ab} is known as the metric, and is extended to form a tensor *field*, acting on vector fields $\vec{v}$ and $\vec{w}$ to give a *differential function* on an open set of a manifold.

With the metric acting on a fixed vector, the combination becomes a $\binom{0}{1}$ type tensor, or a covector:

$$g_{ab}\; v^a \;\equiv\; \zeta_b\;. \tag{1.2.16}$$

This gives a 1-to-1 correspondence between the tangent space and the dual tangent space, assigning a covector to each vector and vice-versa. Since the norm of a vector is also given by the action of its' corresponding covector:

$$(\vec{v})^2 \;=\; g_{ab}\; v^a v^b \;\equiv\; \zeta_b\; v^b\;, \tag{1.2.17}$$

vectors and covectors which are related, are usually given the same name, with the abstract index notation used to tell the vector from the covector:

$$v_b \;\equiv\; g_{ab}\; v^a\;. \tag{1.2.18}$$

Due to this correspondence, the *dual* inner product must be given by the metric inverse:

$$\begin{aligned} && v_b &= g_{ab}\; v^a\;, \\ \Leftrightarrow && (g_{ab})^{-1} v_b &= v^a\;, \\ \Leftrightarrow && (g_{ab})^{-1} v_a v_b &= v_a v^a\;, \\ \Leftrightarrow && (g_{ab})^{-1} v_a v_b &= (v)^2\;, \end{aligned} \tag{1.2.19}$$

with the inverse of the metric, in the abstract index notation, represented simply by raised indices, $g^{ab} := (g_{ab})^{-1}$.

The metric and its inverse can also be used to change whether tensors act on vectors or covectors. In the index notation, the metric is considered to "lower" indices and its inverse to "raise" indices on tensors, vectors and covectors, for example:

$$g_{ab}\, T^{bc} \;=\; T_a^{\;c} \;, \qquad\qquad g^{cd}\, T^a_{\;bc} \;=\; T^{ad}_{\;\;b} \;, \tag{1.2.20}$$

recalling the use of the Einstein summation convention, to sum over each coordinate for the indices b and c above, once a particular coordinate system has been chosen.

Distance Measure

The norm of the infinitesimal distance measure is known as the line element, and is generally denoted by ds^2. In a particular coordinate system, with coordinate tangent vectors dx^i, the line element is given by the sum of the inner product of all combinations of the coordinate vectors:

$$ds^2 \;=\; g_{ij} dx^i dx^j \;, \tag{1.2.21}$$

which can be seen as a generalisation of the Euclidean distance in equation (1.2.3).

For example, in 3-dimensional Euclidean space, with the usual spherical-polar coordinate vectors (r, θ, ϕ), the line element is given by:

$$ds^2 \;=\; dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2 \;, \tag{1.2.22}$$

with the Euclidean metric, in spherical-polar coordinates, given in matrix form by:

$$g_{ab} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \;. \tag{1.2.23}$$

Here, the metric is used to convert the infinitesimal angle changes of $d\theta$ and $d\phi$, into invariant lengths. However, in an arbitrary manifold, the metric also changes the measure of length in different directions, at different points, therefore defining the "shape" of the manifold.

The length of a differential curve in a manifold, can be measured by integrating the metric along the curve. For example, the length of a curve segment γ is found by evaluating the contour integral:

$$l \;=\; \int_\gamma \sqrt{g_{ij} dx^i dx^j} \;, \tag{1.2.24}$$

for any system of coordinate basis vectors x^i. Both the metric, and the length of the curve segment, are coordinate invariant on the manifold. As a result, *any* coordinate system can be used to evaluate the length, all giving the same value for l.

1.2.3 Derivatives on a Manifold

Since “rates of change” with respect to both time and space are so important in physics, it is necessary to have a concept of differentiation with respect to position on a space-time manifold. Hence, it is necessary to define a number of derivatives on an arbitrary differential manifold.

Directional Derivative

A directional derivative of a differential function, at a particular point on a manifold, along the direction of a particular vector field, is defined using the integral curve of that vector field, which passes through the chosen point. The derivative is defined by the ordinary derivative, of the composition of the function with the curve, with respect to the parameter of the curve:

$$D_v f(x) \;:=\; \frac{d}{dt} f\left(\gamma(t)\right)\Big|_{\gamma(t)=x} \;, \tag{1.2.25}$$

where $\gamma(t)$ is the integral curve to the vector field $\vec{v}$, passing through the point $x \in M$. This is then extended to an open set of the manifold, so that the action of the directional derivative is closed on the set of differential functions.

Since the directional derivative is linear in the direction defining vector field, the derivative can be seen as the application of a vector field v^a, to the “gradient” of the function f:

$$D_v f \;\equiv\; v^a\, \partial_a f \;, \qquad \text{s.t.} \qquad \partial_a \;:=\; \frac{\partial}{\partial x^a} \;, \tag{1.2.26}$$

for some system of coordinate vector fields x^a. This leads to the *gradient* being seen as an operator ∂_a, sending differential functions to covector fields.

With the gradient operator understood, the directional derivative can also be seen as the application of a vector field to a differential function. The equivalence class formed by the different results of the directional derivative of a function at a particular point, is sometimes used to define the tangent vectors themselves, at that point.

Covariant Derivative

A covariant derivative is an extension of the concept of a directional derivative, applied to vector, covector and tensor fields. Like the directional derivative, it must be a linear operator, and hence must be linear in both its argument and direction defining vector field, and must follow Leibnitz’ product rule.

With the covariant derivative defined as a linear operator, its action on a vector field, in a particular coordinate system, must be given by:

$$\begin{aligned}\nabla_w\ \vec{v}\ &= \nabla_w \left(v^j \vec{e}_j\right) \\ &= \left(\nabla_w\ v^j\right)\ \vec{e}_j\ +\ v^j\ \left(\nabla_w\ \vec{e}_j\right) \\ &= w^i\ \left(\partial_i\ v^j\right)\ \vec{e}_j\ +\ w^i\ v^j\ \left(\nabla_i\ \vec{e}_j\right)\ . \end{aligned} \tag{1.2.27}$$

Going from the second to third lines above, the first part is due to the coefficients of the vector field being real-valued differential functions, and so the derivative reduces to the ordinary directional derivative. The second part is then due to the linearity of the derivative with respect to $\vec{w}$.

Taking the covariant derivative to be closed on the set of vector fields, the derivative of a vector should also be a vector:

$$w^i\ \nabla_i\ \vec{e}_j\ =\ w^i\ \Gamma^k_{ij}\ \vec{e}_k\ , \tag{1.2.28}$$

with the linear coefficients Γ^c_{ab}, known as the *connection* coefficients. Hence, combining equations (1.2.27) and (1.2.28), the covariant derivative of a vector is given by:

$$\nabla_w\ \vec{v}\ =\ w^i\ \left(\partial_i\ v^j + v^k\ \Gamma^j_{ik}\right)\ \vec{e}_j\ , \tag{1.2.29}$$

for a particular coordinate system, with basis vector fields $\vec{e}_j$.

For Einstein's formulation of general relativity, the covariant derivative is defined to be *torsion-free*, which is equivalent to the connection coefficients being symmetric in the two lower indices:

$$\Gamma^c_{ab}\ =\ \Gamma^c_{ba}\ , \tag{1.2.30}$$

see for example *Misner, Thorne, & Wheeler* [40], for more information.

Since the vector defining the "direction" of the derivative is linear in equation (1.2.29), it can easily be considered as a separate argument for the derivative. This argument can then be separated from the derivative, similar to the directional derivative in equation (1.2.26):

$$\nabla_w\ v^b\ :=\ w^a\ \nabla_a\ v^b\ . \tag{1.2.31}$$

Thus the derivative *operator*, seen as an extension of the *gradient* operator, can be considered as a bilinear operator on vector-fields, denoted in the abstract index notation by ∇_a. From equations (1.2.31) and (1.2.29), the action of the derivative operator on a vector field is given by:

$$\nabla_a\ v^b\ :=\ \partial_a\ v^b\ +\ \Gamma^b_{ac}\ v^c\ , \tag{1.2.32}$$

sending vector fields to type $\begin{pmatrix}1\\1\end{pmatrix}$ tensor fields.

By a similar argument, due to the covariant derivative being a linear operator, and closed on the set of *covector* fields, the action of the derivative operator ∇_a on a covector is given by:

$$\nabla_a \zeta_b \;=\; \partial_a \, \zeta_b \;-\; \Gamma^c_{ab} \, \zeta_c \;, \tag{1.2.33}$$

with Γ^c_{ab} representing the same connection coefficients as (1.2.32)

The covariant derivative can then be extended to tensors of arbitrary type, with the action of the operator ∇_a given by the tensors partial derivative, with the addition of a connection term similar to (1.2.32) for each upper index, and the subtraction of a term similar to (1.2.33) for each lower index. For example:

$$\nabla_a \, T^b_c \;=\; \partial_a \, T^b_c \;+\; \Gamma^b_{ad} \, T^d_c \;-\; \Gamma^d_{ac} \, T^b_d \;, \tag{1.2.34}$$

showing the covariant derivative operator to send type $\binom{m}{n}$ tensors to $\binom{m}{n+1}$ tenors.

The covariant derivative is far from unique, depending on the arbitrary choice of the connection coefficients. With a manifold containing a metric, however, it is natural to choose the derivative so that it commutes with the metrics inner product action. This requires that the covariant derivative of the metric vanish:

$$\nabla_c \, g_{ab} \;=\; 0 \;, \tag{1.2.35}$$

which, combined with the symmetry of Γ^c_{ab} in its two lower indices, see equation (1.2.30), gives an explicit expression for the connection coefficients:

$$\Gamma^c_{ab} \;=\; \frac{1}{2} \, g^{cd} \left(\partial_a \, g_{bd} + \partial_b \, g_{ad} - \partial_d \, g_{ab} \right) \;, \tag{1.2.36}$$

also known, in any specific coordinate system, as *Christoffel* symbols.

With the covariant derivative uniquely defined, tangent spaces at different points can now be related, with the *parallel transport* of a vector v^a, along a curve with tangent vector field u^a, defined by:

$$u^b \, \nabla_b \, v^a \;=\; 0 \;. \tag{1.2.37}$$

This gives a notion of "straight lines" on a curved manifold, with a *geodesic* defined as a curve who's tangent vector is parallel transported along itself:

$$v^a \text{ a geodesic} \qquad \Leftrightarrow \qquad v^b \, \nabla_b \, v^a \;=\; 0 \;. \tag{1.2.38}$$

In a specific coordinate system, the geodesic equation is commonly given in terms of the curves' affine parameter:

$$\frac{\partial^2 x^a}{\partial t^2} \;+\; \Gamma^a_{bc} \, \frac{\partial x^b}{\partial t} \, \frac{\partial x^c}{\partial t} \;=\; 0 \;, \tag{1.2.39}$$

with x^a representing the coordinate basis vectors.

Lie Derivative

The Lie derivative is an alternative way to find a rate of change of a tensor, along different directions in a manifold. However, in order to relate tangent and dual tangent spaces at one point, to those at another point, aside form the use of the covariant derivative, maps between different tangent and cotangent spaces need to be defined.

Vector fields are again used to define the "direction" of the relation between points, by way of their integral curves. A map ϕ_t of differential functions, related to the integral curve γ_t of a vector field v^a, can be defined to return the value of a differentiable function f at the point $\gamma(t)$ to the point $\gamma(0)$:

$$\phi_t\ (f) \ := \ f\left(\gamma(t)\right)\ . \tag{1.2.40}$$

This map is known as a *pull back* of f with respect to the vector field v^a, since the value is "pulled back" from a position further along the integral curve of v^a.

This map can then be extended to vector fields, by their action on differential functions. The vector $\phi_t^*(w)^a$ is thus defined to be that vector which gives the same value when acting on a function f as the original vector w^a gives when acting on $\phi_t(f)$:

$$\phi_t^*\ \left(w|_{\gamma(t)}\right)^a\ (f) \ := \ w^a\left(\phi_t(f)\right)\ |_{\gamma(0)}\ . \tag{1.2.41}$$

The map ϕ^* is known as a *push forward*, since the action of w^a is "pushed forward" to what it would be further along the integral curve of v^a.

Again, this can be extended to covectors, based on their action on vectors:

$$\phi_{*t}\ \left(\zeta|_{\gamma(0)}\right)_a\ w^a \ := \ \zeta_a\ \left(\phi_t^*(w)\right)^a\ |_{\gamma(t)}\ , \tag{1.2.42}$$

which is again known as a *pull back*.

Since ϕ is related to an integral curve, it must be differential, and have a differential inverse. Due to the uniqueness of the actions of covectors on vectors and vectors on differential functions, ϕ must also be bijective. Thus ϕ is a diffeomorphism, and can be extended to tensors with both types of indices, with $(\phi^{-1})^* \equiv \phi_*$.

The Lie derivative can now be defined on an arbitrary tensor T, as the difference between the push forward of T and T itself, in the limit as the parameter $t \to 0$:

$$\mathcal{L}_v\ T \ := \ \lim_{t\to 0}\left(\frac{\phi_t^*\left(T\ |_{\gamma(t)}\right)\ -\ T\ |_{\gamma(0)}}{t}\right) \ = \ \frac{d}{dt}\phi_t^*\left(T\ |_{\gamma(t)}\right)\Big|_{t=0}\ , \tag{1.2.43}$$

which is easily seen as a linear operator, sending any tensor to a tensor of similar type.

For evaluation purposes, the Lie derivative of differential functions, vector fields, covector fields and tensor fields can be given in terms of the covariant derivative.

Applied to differential functions, the Lie derivative reduces to the directional derivative, and is hence also equivalent to the covariant derivative:

$$\mathcal{L}_v\, f \;\equiv\; D_v\, f \;\equiv\; \nabla_v\, f\ . \tag{1.2.44}$$

Choosing coordinates, so that the vector field v^a coincides with the coordinate basis vector field x^1, the Lie derivative of the the vector field w^a, with respect to v^a, reduces from (1.2.43) to:

$$\mathcal{L}_v w^a \;=\; \frac{\partial w^a}{\partial x^1}\ . \tag{1.2.45}$$

The commutator bracket of the vectors v^a and w^a, for a covariant derivative, is defined by:

$$\begin{aligned}[v,w]^a &= v^b \nabla_b\, w^a - w^b \nabla_b\, v^a \\ &= v^b\, \partial_b\, w^a - w^b\, \partial_b\, v^a + \cancel{\Gamma^a_{bd}\, v^b w^d} - \cancel{\Gamma^a_{bd}\, v^d w^b}\ ,\end{aligned} \tag{1.2.46}$$

with the connection coefficients canceling since they are symmetric in the two lower indices, which are both summed over. Again, choosing coordinates so that v^a coincides with the coordinate vector field x^1, the commutator (1.2.46) reduces to:

$$\begin{aligned}[v,w]^a &= v^b\, \partial_b\, w^a - w^b\, \partial_b\, v^a \\ &= x^1\, \partial_{x^1}\, w^a - w^b\, \cancelto{0}{\partial_b\, x^1} \\ &= \frac{\partial w^a}{\partial x^1}\ ,\end{aligned} \tag{1.2.47}$$

which is equivalent to the Lie derivative (1.2.45).

Since both the Lie derivative and commutator bracket are coordinate independent, if they are equivalent for one coordinate system, they must be equivalent for all coordinate systems. Hence the Lie derivative of a vector field w^a with respect to the vector field v^a is given, in general, by their commutator bracket:

$$\begin{aligned}\mathcal{L}_v\, w^a &= [v,w]^a \\ &= v^b\, \nabla_b\, w^a - w^b\, \nabla_b\, v^a \\ &= v^b\, \partial_b\, w^a - w^b\, \partial_b\, v^a\ .\end{aligned} \tag{1.2.48}$$

Thus, by equation (1.2.46) for the commutator bracket, the Lie derivative of a vector field is connection independent, and hence is fully determined, invariant of the metric.

To find an expression for the Lie derivative of a covector field ζ_a, the derivative is first taken of the differential function given by its action on a vector field w^a, and Leibnitz' product rule applied:

$$\mathcal{L}_v \left(\zeta_a w^a\right) \;=\; w^a \; \mathcal{L}_v \zeta_a + \zeta_a \; \mathcal{L}_v w^a \; . \tag{1.2.49}$$

Rearranging, and substituting from (1.2.44) and (1.2.48) for the Lie derivatives of differential functions and vector fields:

$$\begin{aligned} w^a \; \mathcal{L}_v \zeta_a &= \mathcal{L}_v \left(\zeta_a w^a\right) - \zeta_a \; \mathcal{L}_v w^a \\ &= v^b \; \nabla_b \left(\zeta_a w^a\right) - \zeta_a \left(v^b \nabla_b w^a - w^b \nabla_b v^a\right) \\ &= \cancel{\zeta_a v^b \nabla_b w^a} + w^a v^b \nabla_b \zeta_a - \cancel{\zeta_a v^b \nabla_b w^a} + \zeta_a w^b \nabla_b v^a \\ &= w^a v^b \nabla_b \zeta_a + \zeta_a w^b \nabla_b v^a \; , \end{aligned} \tag{1.2.50}$$

and expanding the covariant derivatives again:

$$\begin{aligned} \mathcal{L}_v \zeta_a &= v^b \; \nabla_b \zeta_a + \zeta_b \; \nabla_a v^b \\ &= v^b \; \partial_b \zeta_a + \zeta_b \; \partial_a v^b - \cancel{v^b \; \Gamma^c_{ba} \zeta_c} + \cancel{\zeta_b \; \Gamma^b_{ac} v^c} \; , \end{aligned} \tag{1.2.51}$$

with the canceling due to the two indices b and c being summed over, as well as the connection coefficients being symmetric in the lower two indices. Hence the Lie derivative of a covector field is also connection independent, and invariant of the metric.

The Lie derivative of arbitrary type tensors can each be found by a similar method, using Leibnitz product rule for the Lie derivative of the tensor acting on the appropriate number of vector and covector fields. An inductive method can then show the derivative to be given by the full covariant derivative of the tensor, with a term similar to the second term in (1.2.48) or (1.2.51) for each upper or lower index respectively. As an example, the Lie derivative of a type $\binom{1}{1}$ tensor is given by:

$$\begin{aligned} \mathcal{L}_v \; T^a_b &= v^c \nabla_c \; T^a_b - T^c_b \; \nabla_c \; v^a + T^a_c \; \nabla_b \; v^c \\ &= v^c \partial_c \; T^a_b - T^c_b \; \partial_c \; v^a + T^a_c \; \partial_b \; v^c \; , \end{aligned} \tag{1.2.52}$$

with the connection coefficients from the first term canceling with those of the remaining terms, as with (1.2.46) and (1.2.51). This shows the Lie derivative to be unique, rather than a family of derivatives like the covariant derivative, or being related to the metric.

Taking the special case of the Lie derivative of the metric g_{ab}, using the covariant derivative associated with the metric, giving $\nabla_c \; g_{ab} = 0$, the Lie derivative reduces to:

$$\begin{aligned} \mathcal{L}_v \; g_{ab} &= v^c \cancelto{0}{\nabla_c \; g_{ab}} + g_{cb} \nabla_a \; v^c + g_{ac} \nabla_b \; v^c \\ &= \nabla_a \; v_b + \nabla_b \; v_a \; , \end{aligned} \tag{1.2.53}$$

noting that the covariant derivatives here cannot be reduced to ordinary derivatives.

1.2.4 Coordinate Changes and Curvature

Change of Coordinates

Since space-time is defined to be a differential manifold, geometric objects defined in a chosen coordinate system can be transformed into a different coordinate system by a differential map.

To begin with, a differential function at a point has no dependence on the coordinate basis at that point, and so is completely unaffected by a coordinate choice.

A vector does have different representations depending on the coordinate choice, as seen from equation (1.2.9). Using the chain rule, it can be shown that the relation between a vector v^a in two different coordinate systems x^a and $\bar{x}^b$ is given by:

$$v^a = \frac{\partial x^a}{\partial \bar{x}^b} \, \bar{v}^b \ , \tag{1.2.54}$$

and from this it follows naturally that the relation between the different coordinate representations of a covector ζ_a is given by:

$$\zeta_a = \frac{\partial \bar{x}^b}{\partial x^a} \, \bar{\zeta}_b \ , \tag{1.2.55}$$

with the indices balancing properly, for use of the Einstein summation convention.

The transformation of tensors is naturally given by an appropriate derivative similar to (1.2.54) or (1.2.55) for each upper or lower index respectively. For example:

$$T^a_b = \frac{\partial x^a}{\partial \bar{x}^c} \frac{\partial \bar{x}^d}{\partial x^b} \, \bar{T}^c_d \ , \tag{1.2.56}$$

with the indices again balancing properly.

It is often necessary to carry out coordinate transformations on the metric itself:

$$g_{ab} = \frac{\partial \bar{x}^c}{\partial x^a} \frac{\partial \bar{x}^d}{\partial x^b} \, \bar{g}_{cd} \ , \tag{1.2.57}$$

which simply transforms similar to any type $\binom{0}{2}$ tensor.

Tensor Densities

Tensor densities are tensor-type functionals which, due to a dependence on the metric determinant, do not change in the usual way under coordinate transformations.

This can be seen most clearly, by taking equation (1.2.57) for the coordinate transformation of the metric, and taking the determinant of both sides:

$$det\,|g_{ab}| \;=\; det\left|\frac{\partial\bar{x}^c}{\partial x^a}\frac{\partial\bar{x}^d}{\partial x^b}\right|\; det\,|\bar{g}_{cd}|\ , \tag{1.2.58}$$

giving the coordinate transformation of the determinant of the metric, denoted by g:

$$g \;=\; det\left|\frac{\partial\bar{x}^c}{\partial x^a}\frac{\partial\bar{x}^d}{\partial x^b}\right|\; \bar{g}\ . \tag{1.2.59}$$

The determinant g is a real valued differential function, and as such, would be expected to be coordinate invariant, however here it contains a transformation factor J:

$$J \;=\; det\left|\frac{\partial\bar{x}^c}{\partial x^a}\frac{\partial\bar{x}^d}{\partial x^b}\right|\ , \tag{1.2.60}$$

which is known as the Jacobian.

As a result of this transformation, any tensor that depends on the metric determinant g, must also alter its coordinate transformation. Such tensors are known as *tensor densities* and transform with an extra factor, consisting of a power of the Jacobian. For example, a $\binom{1}{1}$ type tensor *density*, would transform according to:

$$\mathcal{T}^a_b \;=\; J^W\,\frac{\partial x^a}{\partial\bar{x}^c}\frac{\partial\bar{x}^d}{\partial x^b}\;\mathcal{T}^c_d\ , \tag{1.2.61}$$

where J is the Jacobian, and W is known as the "weight" of the tensor density, generally twice the power of the metric determinant factor.

The covariant derivative of a tensor density can easily be shown to be given by the usual covariant derivative terms, with the subtraction of an extra term:

$$\nabla_c\,\mathcal{T} \;=\; \nabla_c\,T\ - W\,\Gamma^d_{dc}\,\mathcal{T}\ , \tag{1.2.62}$$

with $\nabla_c T$ representing the usual covariant derivative for that type of tensor. The *Lie* derivative of a tensor density can then be found directly from its covariant derivative, by use of equations (1.2.48) to (1.2.52), giving:

$$\mathcal{L}_v\,\mathcal{T} \;=\; \mathcal{L}_v\,T\ + W\,\mathcal{T}\,\nabla_c v^c\ , \tag{1.2.63}$$

with $\mathcal{L}_v T$ again representing the ordinary tensor Lie derivative. For a $\binom{1}{1}$ tensor density, for example, the covariant and Lie derivatives are given by:

$$\nabla_c\,\mathcal{T}^a_b \;= \partial_c\,\mathcal{T}^a_b + \Gamma^a_{cd}\,\mathcal{T}^d_b - \Gamma^d_{cb}\,\mathcal{T}^a_d - W\,\Gamma^d_{dc}\,\mathcal{T}^a_b\ , \tag{1.2.64}$$

$$\mathcal{L}_v\,\mathcal{T}^a_b \;= v^c\nabla_c\,\mathcal{T}^a_b - \mathcal{T}^c_b\,\nabla_c\,v^a + \mathcal{T}^a_c\,\nabla_b\,v^c + W\,\mathcal{T}^a_b\,\nabla_c v^c\ . \tag{1.2.65}$$

Curvature Tensors

A manifold which is embedded in a Euclidean space of a higher dimension has a natural concept of curvature, for example, a flat table surface, or the curved surface of a ball, embedded in the 3-dimensional space of a room. However, in relativity theory, space-time is not thought of as embedded in any higher dimensional space, hence an "intrinsic" notion of curvature is needed.

Using the parallel transport of a vector, from equation (1.2.37), it can be seen that any closed path in a flat manifold will leave the vector unchanged. However, the parallel transport along a closed path on the surface of a sphere, does not leave a vector unchanged, as can be seen in figure 1.2.

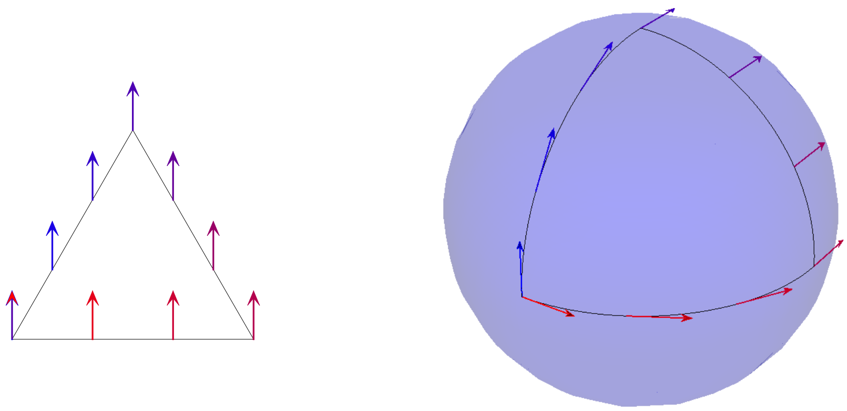


Figure 1.2: The change in a vector, parallel transported along a closed path, on both a Euclidean surface, and on the surface of a sphere.

This can also be seen as the triangle inequality holding for a "flat" manifold, but not for the surface of the sphere, with the example in figure 1.2 containing *three* right angles instead of summing to two.

Since the change in a vector is not the same for all closed paths, a precise definition of intrinsic curvature requires the path to be shrunk to an infinitesimal size. The *Riemann* curvature tensor is then defined as the adjustment to a vector, to give the action of the commutator of two covariant derivatives, on that vector:

$$\nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c \;=\; v_d \; R^d{}_{cab} \;, \tag{1.2.66}$$

which gives a 4-index tensor, of type $\binom{1}{3}$, though this can be changed by application of the metric, or its inverse.

The Riemann tensor, with all indices lowered, is symmetric and antisymmetric under swapping of the following index positions:

$$R_{abcd} \;=\; -R_{bacd} \;=\; -R_{abdc} \;=\; R_{cdab} \;. \tag{1.2.67}$$

The Riemann curvature also satisfies both the cyclic and Bianchi identities:

$$R_{abcd} + R_{adbc} + R_{acdb} \;= 0 \;, \tag{1.2.68}$$

$$\nabla_e \; R_{abcd} + \nabla_d \; R_{abec} + \nabla_c \; R_{abde} \;= 0 \;. \tag{1.2.69}$$

In a given coordinate system, the Riemann curvature tensor can be calculated by use of the connection coefficients (1.2.36), using the equation:

$$R^a{}_{bcd} \;=\; \partial_c \; \Gamma^a_{bd} - \partial_d \; \Gamma^a_{bc} + \Gamma^a_{ec} \; \Gamma^e_{bd} - \Gamma^a_{ed} \; \Gamma^e_{bc} \;, \tag{1.2.70}$$

in the form given by *Misner, Thorne & Wheeler* [40]. Since the connection coefficients are themselves given by first order derivatives of the metric, from the first two terms above, the Riemann curvature can be seen to be given by *second* order derivatives of the metric.

The identities given in equations (1.2.67), show the only non-zero contractions of the Riemann curvature tensor to be given by the tensor:

$$R_{ab} \;=\; R^c{}_{acb} \;, \tag{1.2.71}$$

which is known as the *Ricci* curvature tensor. The Ricci tensor can also be seen, from equations (1.2.67), to be symmetric in its two indices.

The trace of the Ricci curvature then gives what is known as the *scalar* curvature:

$$R \;=\; R^a_a \;=\; g^{ab} R_{ab} \;. \tag{1.2.72}$$

Since both the Ricci and scalar curvatures are given by contractions of the Riemann curvature tensor, from equation (1.2.70) they can also be determined completely by the metric, along with its first and second order derivatives.

1.2.5 Killing Vectors and the Frobenius Theorem

Killing Vector Fields

Killing vector fields provide a symmetry of the space-time manifold, by preserving the metric along their path. This metric preservation is defined by having the Lie derivative of the metric vanish, when taken with respect to a Killing vector field:

$$\vec{v} \text{ a Killing vector field} \qquad \Leftrightarrow \qquad \mathcal{L}_v \, g_{ab} \;=\; 0 \; . \tag{1.2.73}$$

From equation (1.2.53), a vector field v^a is a Killing field if and only if it satisfies:

$$\nabla_a \, v_b + \nabla_b \, v_a \;=\; 0 \; , \tag{1.2.74}$$

which is known as *Killings* equation.

Since the covariant derivative is a linear operator, any linear combination of Killing vectors is also a Killing vector. It can also be shown that the commutator bracket of two Killing vectors v^a and w^a, is also a Killing vector, satisfying equation (1.2.74):

$$\begin{aligned}
\nabla_a[v,w]_b + \nabla_b[v,w]_a \;&=\; \nabla_a \left(v^d \nabla_d w_b - w^d \nabla_d v_b \right) + \nabla_b \left(v^d \nabla_d w_a - w^d \nabla_d v_a \right) \\[2ex]
&= v^d \nabla_d \nabla_a w_b + v^d w_c R^c_{bad} + v^d \nabla_d \nabla_b w_a + v^d w_c R^c_{abd} \\
&\quad - w^d \nabla_d \nabla_a v_b - w^d v_c R^c_{bad} - w^d \nabla_d \nabla_b v_a - w^d v_c R^c_{abd} \\
&\quad - \nabla_a v^d \cdot \nabla_b w_d - \nabla_b v^d \cdot \nabla_a w_d + \nabla_a w^d \cdot \nabla_b v_d + \nabla_b w^d \cdot \nabla_a v_d \\[2ex]
&= v^d \nabla_d \cancelto{0}{\left(\nabla_a w_b + \nabla_b w_a \right)} - w^d \nabla_d \cancelto{0}{\left(\nabla_a v_b - \nabla_b v_a \right)} \\
&\quad + v^d w_c R^c_{bad} + v^d w_c R^c_{abd} - w^d v_c R^c_{bad} - w^d v_c R^c_{abd} \\
&\quad - \nabla_a v_d \cdot \nabla_b w^d - \nabla_b v_d \cdot \nabla_a w^d + \nabla_a w^d \cdot \nabla_b v_d + \nabla_b w^d \cdot \nabla_a v_d \\[2ex]
&= v^d w^c R_{cbad} - v^c w^d R_{cbad} + v^d w^c R_{cabd} - v^c w^d R_{cabd} \\
&\quad - \cancel{\nabla_a v_d \cdot \nabla_b w^d} + \cancel{\nabla_b w^d \cdot \nabla_a v_d} \;\; - \cancel{\nabla_b v_d \cdot \nabla_a w^d} + \cancel{\nabla_a w^d \cdot \nabla_b v_d} \\[2ex]
&= v^c w^d \left(R_{dbac} - R_{cabd} + R_{dabc} - R_{cbad} \right) \;=\; 0 \; ,
\end{aligned} \tag{1.2.75}$$

using the definition of the Riemann tensor (1.2.66) in the second line, along with the derivative being invariant of the metric. Killings equation (1.2.74) is then used to cancel the brackets in the third line, and finally, the Riemann tensor index changes in (1.2.67), show the final line to vanish. Thus, the vector space of Killing vectors, a subspace of the tangent space at each point, is closed under the commutator bracket.

Frobenius' Theorem

The Frobenius theorem gives the necessary and sufficient conditions for the existence of integral submanifolds, and the conditions under which a manifold can be "foliated" into a continuous family of integral submanifolds.

The *span* of a set of tangent vectors $\{(v_i)^a\}$ at a point x on a manifold, is defined to be the set of all vectors which can be expressed by linear combinations of $(v_i)^a$, thus giving a vector subspace of the tangent vector space at the point x. This can then be extended to the span of vector fields $(v_i)^a$:

$$\mathrm{span}\{(v_i)^a\} \;:=\; \left\{ v^b \;\middle|\; v^b = \sum_i \,(C_i)\;(v_i)^a \right\}, \tag{1.2.76}$$

for (C_i) differential functions. The span of a set of vector fields can similarly be seen to define a vector subspace of the tangent space at each point of an open set on which the span is defined.

Theorem 1 (Frobenius' Theorem)**.**

A span of vector fields V, which gives an m-dimensional vector subspace at each point of a manifold M, defines a foliation of M into a family of m-dimensional integral submanifolds, if and only if, V is closed under the commutator bracket.

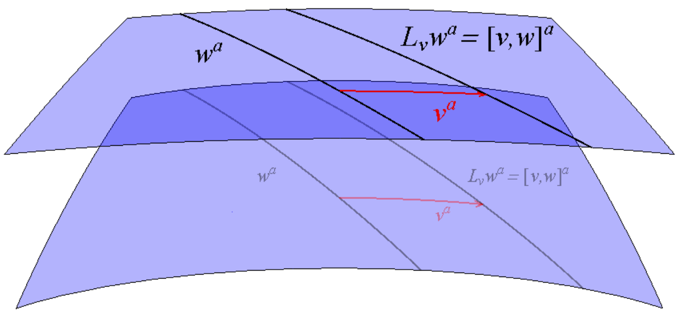


Figure 1.3: Two integral manifolds, with an integral curve of w^a Lie dragged along v^a to define a family of integral submanifolds.

As a direct result of theorem 1, since the set of Killing vector fields is closed under the commutator bracket, by (1.2.75), then on a manifold the set of Killing vectors defines a foliation of the manifold into integral submanifolds, with the Killing vectors as tangent vectors.

The Frobenius theorem can also be formulated in a dual form for covector fields. Firstly, it is well known from linear algebra, that for every subspace V_x of tangent vectors at a point x, there exists a dual subspace S^*_x such that:

$$\zeta_a \; v^a \; = \; 0 \qquad \forall \quad \zeta_a \in S^*_x, \quad v^a \in V_x \; , \tag{1.2.77}$$

and vice-versa, with the dimensions of the two subspaces adding to that of the manifold. This can be extended in a natural way to spans of vector and covector *fields* V and S^*.

By theorem 1, a manifold is therefore foliated into a family of integral submanifolds, if there exists a span of covector fields S^*, such that:

$$\zeta_a \; v^a \; = \; \zeta_a \; w^a \; = \; 0 \tag{1.2.78}$$

$$\Rightarrow \qquad \zeta_a \; [v, w]^a \; = \; 0, \qquad\qquad \forall \; \zeta_a \in S^* \; . \tag{1.2.79}$$

Expanding the commutator bracket in (1.2.79), by the covariant derivative form of equation (1.2.46):

$$\begin{aligned} 0 \; &= \zeta_a \; [v, w]^a \\ &= \zeta_a \left(v^b \; \nabla_b \; w^a - w^b \; \nabla_b \; v^a \right) \\ &= v^b \; \left(\nabla_b \; (\cancelto{0}{\zeta_a w^a}) - w^a \; \nabla_b \; \zeta_a \right) \\ &\quad - w^b \; \left(\nabla_b \; (\cancelto{0}{\zeta_a v^a}) - v^a \; \nabla_b \; \zeta_a \right) \\ &= v^a \; w^b \; \left(\nabla_b \; \zeta_a - \nabla_a \; \zeta_b \right) \; , \end{aligned} \tag{1.2.80}$$

with the canceling in the third line due to the vector fields v^a and w^a satisfying (1.2.78).

The final part of (1.2.80) is true for vector fields v^a and w^a satisfying (1.2.78), if and only if the term in the brackets is given by a linear combination of covector fields in S^*, i.e. for all $\zeta_a \in S^*$:

$$\nabla_b \; \zeta_a - \nabla_a \; \zeta_b \; = \; \sum_i \; \Big((\zeta_i)_a \; (\xi_i)_b - (\zeta_i)_b \; (\xi_i)_a \Big), \quad \text{s.t.} \quad (\zeta_i)_a \in S^* \; , \tag{1.2.81}$$

with $(\xi_i)_b$ representing *arbitrary* covector feilds. The subtraction inside the brackets is required to keep the right side of the equation antisymmetric, in agreement with the left side. The dual form of the theorem 1 can now be given.

Theorem 2 (Frobenius' Theorem - Dual Form)**.**

A span of covector fields S^*, giving an s-dimensional covector subspace at each point of an n-dimensional manifold M, defines a foliation of M into a family of $(n-s)$-dimensional integral submanifolds, if and only if, for all $\zeta_a \in S^*$:

$$\nabla_b \, \zeta_a - \nabla_a \, \zeta_b \;=\; \sum_i \Big((\zeta_i)_a \; (\xi_i)_b - (\zeta_i)_b \; (\xi_i)_a \Big), \qquad s.t. \quad (\zeta_i)_a \in S^* \;, \tag{1.2.82}$$

for arbitrary covector fields $(\xi_i)_b$.

Proofs of the Frobenius theorem can be found in *Chevalley* [18] (Chapter 3, Sections VI - VIII), where the theorem was first formulated in geometric terms, and in *Ivey & Landsberg* [32] (page 30), which proves theorem 2 using exterior calculus of differential forms.

1.3 General Relativity

1.3.1 The Special Theory of Relativity

In Galilean relativity, an observer which experiences no acceleration, known as an *inertial* observer, can set up a 3-dimensional Euclidean frame of reference. At any point in time, the coordinates of one "inertial frame", can be translated into those of another, according to the relative velocity between the two. As such, intervals in time, and measurements in space, are considered to be the same for all inertial observers.

With Galileo's relative velocity, the speed of light c measured by two inertial observers A and B, with a relative velocity of v, are related by the equation:

$$c_A \;=\; v + c_B \;. \tag{1.3.1}$$

However, more and more evidence in the later half of the 1800s, due in particular to Maxwell's theory of electromagnetism, and the experiments of Michelson and Morely, showed that the speed of light should be *constant* for all observers. Hence the speed of light measured by A and B above should give $c_A = c_B$.

Einstein showed, with his *special* theory of relativity, that the apparent inconsistencies of Galileo's relativity, and the constancy of the speed of light, could be overcome if different inertial observers found the *time* intervals between two events to be *different*, also implying that *spatial* distances would be measured differently.

The relation between the time and space intervals of two inertial frames L and L', moving with relative velocity v in direction x, are given by the Lorentz transformations:

$$\begin{aligned} \Delta t' &= \gamma(\Delta t - \Delta x/c^2) \;, \\ \Delta x' &= \gamma(\Delta x - v\Delta t) \;, \qquad \text{with} \qquad \gamma \;=\; (1 - v^2/c^2)^{-1/2} \;, \end{aligned} \tag{1.3.2}$$

with the term γ is known as the Lorentz factor. Investigation of the Lorentz factor, shows that an observer traveling at the speed of light experiences *no* passing of time, and an observer traveling *faster* than the speed of light experiences a *reversing* of time. Hence, the speed of light is seen as a "limiting" speed for all objects.

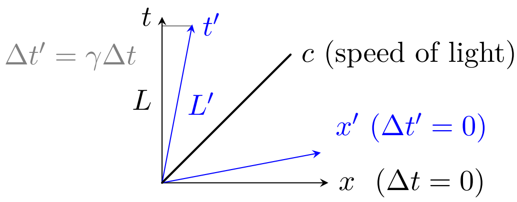


Figure 1.4: Two inertial frames, showing different "surfaces of constant time" for each.

Though special relativity removes the invariance of both time and space intervals, for a light signal passing through two different events, each inertial observer will measure their own time interval Δt, and spatial distance Δd, between the two events. Since the speed of light is the same for all inertial frames, each observer will find:

$$\frac{(\Delta d)^2}{(\Delta t)^2} \;=\; c^2 \qquad\Leftrightarrow\qquad 0 \;=\; -(c\Delta t)^2 + (\Delta d)^2 \ , \tag{1.3.3}$$

showing this new Euclidean-type "space-time" interval, to be zero for *all* events with a "light-like" separation, for *all* inertial observers.

Based on the work of Henri Pioncaré, in 1908 Hermann Minkowski was able to show, that extending the interval given by equation (1.3.3), to *any* two events:

$$(\Delta s)^2 \;:=\; -(c\Delta t)^2 + (\Delta d)^2 \ , \tag{1.3.4}$$

is also invariant for all inertial observers. The choice of sign for the interval $(\Delta s)^2$, agreeing with that of the spatial interval, is convention. As such, a negative interval $(\Delta s)^2 < 0$, represents a separation which can be traveled with a speed less than that of light, with a positive interval $(\Delta s)^2 > 0$, incapable of being traversed by any object.

With space-time considered to form a 4-dimensional differential manifold, the negative square of the time interval can be obtained by either taking "time-like" vectors to be imaginary, or assuming a non-positive-definite metric. The later is taken as convention, which along with the convention above, agrees with *M. T. & W.* [40], *H. & E.* [31] and *Wald* [49]. The infinitesimal form of the interval (1.3.4), is then given by:

$$ds^2 \;=\; \eta_{\mu\nu} dx^\mu dx^\nu \ , \tag{1.3.5}$$

with the metric η_{ab} known as the *Minkowski* metric. To give the interval in (1.3.4), the matrix representation, in Cartesian type coordinates, must take the form:

$$\eta_{\mu\nu} \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{1.3.6}$$

using "geometrized units", defined by setting the speed of light $c := 1$, with the $(-{+}{+}{+})$ signs of the components, known as a "Lorentz" signature. Greek indices are specifically used from here on, to represent the *four* dimensions of *space-time* manifolds.

The Minkowski metric $\eta_{\mu\nu}$, in Cartesian coordinates, has vanishing connection coefficients, by equation (1.2.36), giving the associated covariant derivative operator as the ordinary gradient operator ∂_μ. The curvature tensors are also zero, in *all* coordinate systems, resulting in the Minkowski metric seen as describing a "flat" space-time.

1.3.2 The General Theory of Relativity

General Space-Time Manifold

From the principle of equivalence, outlined in section 1.1, space and time in the *general* theory of relativity, are considered to form a non-Euclidean 4-dimensional differential manifold, with each point representing an "event" in space and time.

In order to maintain the time and space relations given by special relativity, a *general* space-time metric must also have a Lorentz signature. Since a general metric will not necessarily have the simplified form of the Minkowski metric, it is the four *eigenvalues* of any matrix form of the metric, which must have signs $(-+++)$.

The Lorentz signature of the metric gives a non-positive-definite line element, similar to equation (1.3.4), providing an important distinction in the separation of events:

$$\begin{aligned} ds^2 &< 0 : \quad \text{separation is time-like,} \\ ds^2 &= 0 : \quad \text{separation is null,} \\ ds^2 &> 0 : \quad \text{separation is space-like.} \end{aligned} \tag{1.3.7}$$

Along with the limiting speed of light, this leads to the concept of causality in relativity theory, and the ambiguity in the concept of simultaneity. An event with a "time-like" separation from another, can either influence that event or be influenced by it, while any pair of events with a "space-like" separation can be considered to happen *simultaneously*, by different observers.

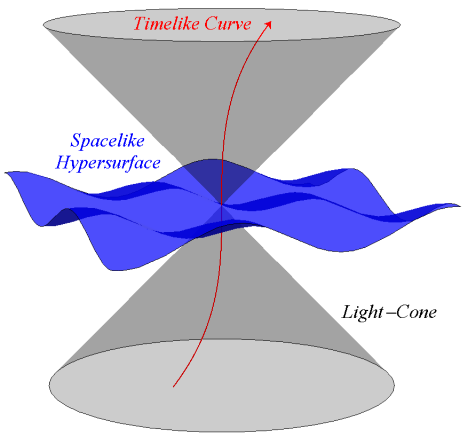


Figure 1.5: The light cone of an event, with a space-like hypersurface containing it, and time-like curve passing through it. (Note one spatial dimension suppressed).

For a general Lorentz metric, a "space-like" hypersurface can be given by *any* 3-dimensional submanifold of the 4-dimensional space-time manifold, which doesn't cross the light cone of any of its elements, see figure 1.5 above. The events contained by such a hypersurface, can all be considered to happen "at the same time" by some observer.

Stress-Energy Tensor

So far, all of the geometry needed to describe the shape of space-time has been introduced. It remains, however, to describe the "mass" that is to shape this space-time, according to Mach's principle. It is worth noting, that the mass-energy equivalence in Einstein's *special* theory of relativity:

$$E = M\, c^2 \,, \tag{1.3.8}$$

which, in geometrized units is simply $E = M$, implies that *all* energy sources need to be taken into account for the gravitational influence on the space-time geometry.

The mass-energy density turns out to be given by a type $\binom{0}{2}$ tensor, called the *stress-energy* tensor, and denoted by $T_{\mu\nu}$. The components of the stress-energy tensor, in the coordinates of any inertial frame, can be interpreted as:

$$\begin{aligned} T_{00} &= \text{energy density,} \\ T_{0i} &= \text{momentum density,} \\ T_{ij} &= \text{flux of } i \text{ momentum in } j \text{ direction,} \end{aligned} \tag{1.3.9}$$

with the 0 index representing the "time" coordinate vector field, and i and j representing the 3 "spatial" coordinates.

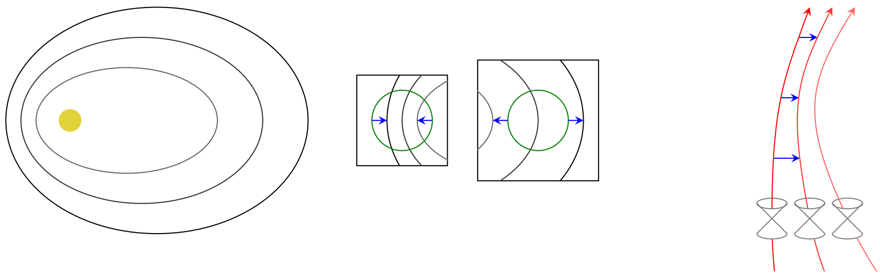
With the "geometrized units" introduced earlier, the gravitational constant G is also defined to be unitary, so $G = c = 1$.

Einstein's Field Equations

Returning to the principle of equivalence, regardless of the presence of any gravitational field, observers in "free-fall", for example astronauts orbiting the Earth, *experience* zero acceleration. Hence, in any space-time manifold, there will always exist *local* inertial frames, which follow time-like geodesics in the manifold.

In each of these "local" inertial frames, a specific choice of coordinates can be made, giving the metric in the form of the Minkowski metric. However, in the presence of a gravitational field, each inertial frame will accelerate with respect to others, and so a coordinate system which provides a Minkowski metric in one inertial frame, will not necessarily give a Minkowski metric in another. Hence, for general relativity to apply to *all* frames, it must be formulated in a coordinate-*free* or "generally covariant" form.

Since local inertial frames experience no acceleration, along any time-like geodesic, there can be no gravitational "force" experienced, regardless of the curvature of the manifold. However, the distance between nearby geodesics *will* change, as a result of curvature. This is equivalent to the "tidal" force experienced by large orbiting objects, with different parts of the object requiring slightly different "free-fall" paths.



(a) Elliptic orbits around a mass. (b) Tidal acceleration. (c) Deviation of time-like geodesics.

Figure 1.6: Relating the Newtonian tidal force to the deviation of time-like geodesics.

Since Newton's gravitational force is a conservative force, it can be given by the gradient of a potential energy V. To give the gravitational force as a *field*, in a 3-dimensional Euclidean space, the mass of any particles in the field are factored out of V, giving a *gravitational* potential Φ:

$$\Phi \;:=\; V/m \qquad \Rightarrow \qquad F \;=\; m\, a \;=\; -m\, \nabla\Phi\ , \tag{1.3.10}$$

with ∇ representing the Euclidean gradient operator here. For a region with a mass density of ρ, the gravitational potential can be found from the equation:

$$\partial_i \partial^i \Phi \;=\; \nabla^2 \Phi \;\equiv\; 4\pi G \rho \;=\; 4\pi\rho\ , \tag{1.3.11}$$

recalling that $G = 1$ in geometrized units. This gives a Newtonian analogue for the "00" part of the stress-energy tensor, in the coordinate system of any inertial observer.

The Newtonian tidal effect between two points, is given by the difference in the accelerations due to the gravitational potential at each point. Infinitesimally, this difference is given by the directional derivative of the acceleration. With equation (1.3.10) giving an expression for the acceleration in terms of the gravitational potential, the directional derivative, with respect to a vector field s^i, is applied to both sides:

$$D_s\, a \;= D_s \nabla\Phi\ ,$$

$$\begin{aligned} \Leftrightarrow \qquad D_s\, \partial_t^2 x^i \;&= s^j \partial_j\, \partial^i \Phi\ , \\ \Leftrightarrow \qquad \partial_t^2\, D_s x^i \;&= s^j \partial_j \partial^i \Phi\ , \\ \Leftrightarrow \qquad \partial_t^2 s^i \;&= s^j\, \partial_j \partial^i \Phi\ , \end{aligned} \tag{1.3.12}$$

using the equivalence of mixed derivatives for the third line, and noting for the final line, that the rate of change of *position* along any tangent vector, *is* that vector.

The tidal acceleration can be seen in figures 1.6(a) and 1.6(b) to be equivalent to the *relative* acceleration of nearby test particles. In general relativity, test particles follow time-like geodesics, and so the tidal acceleration is given by the relative acceleration of nearby time-like geodesics, see figure 1.6(c).

To find an expression for the deviation of nearby geodesics, a smooth one-parameter family of time-like geodesics γ_s is first taken, with each geodesic an integral curve of a vector field u^μ. The *deviation* vector field s^μ, gives the displacement between two infinitesimally close elements of the family, such that s^μ and u^μ can be seen as orthogonal, along each geodesic in γ_s.

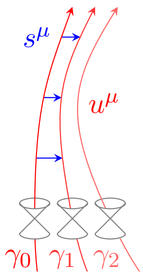


Figure 1.7: Deviation of time-like geodesics.

The relative acceleration of two infinitesimally close geodesics in γ_s can be given by the acceleration of s^μ, with respect to the geodesic tangent vectors u^μ:

$$a^\kappa \;=\; \nabla_u \nabla_u \; s^\kappa \;=\; u^\mu \nabla_\mu \; u^\nu \nabla_\nu \; s^\kappa \;, \tag{1.3.13}$$

with ∇_μ again representing the *covariant* derivative operator. From the relationship between s^μ and u^μ, the geodesic equation (1.2.38) and the definition of the Riemann curvature tensor (1.2.66), the *geodesic deviation* can be shown to be given by:

$$u^\mu \nabla_\mu \; u^\nu \nabla_\nu \; s^\kappa \;=\; -R_{\mu\lambda\nu}{}^\kappa \; s^\lambda \; u^\mu u^\nu \;, \tag{1.3.14}$$

relating the infinitesimal deviation of nearby geodesics, directly to the intrinsic curvature of the manifold. See e.g. *Wald* [49] (section 3.3) for a full derivation.

Since inertial frames follow time-like geodesics, the vector field u^μ for the geodesics in γ_s, gives the "time" vectors for any associated inertial frames. The acceleration in equation (1.3.13), can therefore be seen as the second *time* derivative of the space-like vector field s^μ, relating the relative acceleration of two infinitesimally close observers, corresponding with the vector field s^i in equation (1.3.12). With the "energy density" measured by such an observer given by $T_{\mu\nu}u^\mu u^\nu$, the correspondence:

$$\partial_t^2 \; s^i \quad \longrightarrow \quad u^\mu \nabla_\mu \; u^\nu \nabla_\nu \; s^\kappa \;, \tag{1.3.15a}$$

$$\rho \quad \longrightarrow \quad T_{\mu\nu} u^\mu u^\nu \;, \tag{1.3.15b}$$

from Newtonian gravity, to a relativistic space-time manifold, seems plausible.

The correspondences (1.3.15), along with the geodesic deviation equation (1.3.14), and the Newtonian equations (1.3.12) and (1.3.11), then imply the field equations:

$$\begin{aligned} & R_{\mu\lambda\nu}{}^{\kappa}\ u^{\mu}u^{\nu} &= 4\pi\ T_{\mu\nu}\ u^{\mu}u^{\nu}\ , \\ \Leftrightarrow\qquad & R_{\mu\nu} &= 4\pi T_{\mu\nu}\ , \end{aligned} \tag{1.3.16}$$

which were themselves hypothesized by Einstein in 1913. However, independently, both Einstein and David Hilbert found the equations above to be flawed. In Hilbert's case, a variational method showed the only self consistent equations to be given instead by:

$$G_{\mu\nu} \ = \ R_{\mu\nu} - \frac{1}{2}\ R\ g_{\mu\nu} \ = \ 8\pi\ T_{\mu\nu}\ , \tag{1.3.17}$$

with the tensor $G_{\mu\nu}$ known as the Einstein or Hilbert tensor. The equations above are known as the Einstein Field Equations, and provide the final coupling of the space-time geometry to the total distribution of mass-energy, in accordance with Mach's principle.

The Einstein-Hilbert tensor can also be seen, from the Bianchi identities of equation (1.2.69), to be divergence-free, implying the same for the stress-energy tensor:

$$\nabla_{\mu}G^{\mu\nu} \ = \ 0 \qquad \Leftrightarrow \qquad \nabla_{\mu}T^{\mu\nu} \ = \ 0\ , \tag{1.3.18}$$

providing a *local* concept of mass-energy conservation for general relativity.

The field equations can also be rewritten, with the trace of the stress-energy tensor used instead of the scalar curvature:

$$\begin{aligned} & R_{\mu\nu} \ - \ \frac{1}{2}Rg_{\mu\nu} &= \ 8\pi T_{\mu\nu}\ , \\ \Leftrightarrow\qquad & R_{\mu\nu} &= \ 8\pi T_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu}\ , \\ \Leftrightarrow\qquad & R_{\mu\nu} &= \ 8\pi\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)\ , \end{aligned} \tag{1.3.19}$$

since the trace of equation (1.3.17) gives $R = -8\pi T$, where $T = g^{\mu\nu}T_{\mu\nu}$. For a vacuum solution ($T_{\mu\nu} = 0$), this form agrees with the earlier hypothesized equations (1.3.16).

Since $G_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric 4-dimensional tensors, there are 10 separate components in each. The Ricci and scalar curvatures are then given by second order expressions for the metric, giving the field equations (1.3.17), as a set of 10 coupled, second order, partial differential equations for the metric. Due to the complexity of this system, only a few exact solutions are known, all of which have high degrees of symmetry. The only other method of finding solutions comes from numerically solving the equations, requiring specific re-formulations.

Although there are 10 field equations for the 10 metric components, 4 of these components are given by arbitrary coordinate choices. This would reduce the field equations to 6, however there is no "natural" way of separating these 6 from the 10.

1.3.3 Schwarzschild Solution

As mentioned above, the only exact solutions of Einstein's field equations (1.3.17) feature high degrees of symmetry, which are used to decouple and simplify the equations.

The first such solution was put forward by Karl Schwarzschild, just a few months after Einstein presented his theory. Schwarzschild published two papers in January and February of 1916, the first [44] (see English translation *Antoci & Loinger* [3]) is a vacuum solution for a point mass, and the second [45] (see English translation *Antoci* [2]), a non-vacuum solution for a spherical, incompressible fluid.

Derivation of Schwarzschild Vacuum Solution

The Schwarzschild "point mass" solution, assumes a static, spherically symmetric, vacuum space-time.

Since the space-time is static, it must contain a time-like Killing vector. If the time coordinate is chosen to coincide with this Killing vector, then there can be no dependence of the metric on the time coordinate, and there will be no interaction between the "time" and "space" parts of the metric. Hence, with this choice of coordinates, the Schwarzschild metric is given, in matrix form, by:

$$g_{\mu\nu} = \begin{pmatrix} -f(x) & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \gamma_{ij}(x) & \\ 0 & & & \end{pmatrix} , \tag{1.3.20}$$

with the indices i and j running across the 3 spatial coordinates alone, represented by x, and $f(x)$ a positive differential function. The 3×3 matrix, representing the "spatial" part of the metric, is denoted by $\gamma_{ij}(x)$.

The property of spherical symmetry implies that there must exist a two dimensional vector space of Killing vectors, at each point in the spatial submanifold. By Frobenius' theorem 1, this space of Killing vectors foliates the spatial metric into a family of spherical surfaces. Since each sphere can be completely characterized by its area, a "radial" coordinate can be defined, so that:

$$A = 4\pi r^2 , \qquad \rightarrow \qquad r := \left| \left(\frac{A}{4\pi} \right)^{1/2} \right| , \tag{1.3.21}$$

with the metric entirely determined by this coordinate r, known as the *areal radius*.

The spatial part of the metric can now be given in terms of the radial coordinate r, in a spherical-polar type coordinate system $(r,\ \theta,\ \phi)$, see equation (1.2.23):

$$\gamma_{ij} \ = \ \begin{pmatrix} h(r) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} , \tag{1.3.22}$$

with the "rr" component given by a positive differential function $h(r)$, since the infinitesimal distance measure *between* neighbouring spherical surfaces is unknown.

Since the metric is invariant to changes in the θ and ϕ coordinates (except for the inclinational variance of the "$\phi\phi$" term), the function $f(x)$ for the "tt" term can depend *only* on the radial coordinate r. Combining equations (1.3.20) and (1.3.22), then gives:

$$g_{\mu\nu} \ = \ \begin{pmatrix} -f(r) & 0 & 0 & 0 \\ 0 & h(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} . \tag{1.3.23}$$

This form of the metric now consists of 2 functions of 1 coordinate variable, instead of the general 10 functions of 4 coordinate variables.

The Ricci curvature tensor, for a metric of the form (1.3.23), can also be shown using (1.2.71) and (1.2.70) to be diagonal. Hence, by the vacuum assumption, giving the stress energy tensor $T_{\mu\nu} = 0$, Einstein's equations in the from of (1.3.19), become:

$$R_{\mu\mu} \ = \ 0 \ , \tag{1.3.24}$$

reducing to *ordinary* differential equations in r. The solutions to (1.3.24) take the form:

$$f(r) \ \equiv \ \left(1 + \frac{A}{r}\right) , \qquad h(r) \ \equiv \ \left(1 + \frac{A}{r}\right)^{-1} , \tag{1.3.25}$$

for some constant A (see e.g. *Wald* [49], *Alcubierre* [1]). By comparing with Newtonian gravity, substituting for empirical observation, the constant A is given by:

$$A \ := \ -2M \ , \tag{1.3.26}$$

with M representing the mass measured at an infinite distance (the geometric Newtonian limit). The full Schwarzschild solution is thus given by the metric:

$$g_{\mu\nu} \ = \ \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} , \tag{1.3.27}$$

in terms of the *Schwarzschild coordinates* (t, r, θ, ϕ) and the *Schwarzschild mass* M.

It is obvious from equation (1.3.27), that in the limiting case of zero mass, the Schwarzschild metric reduces to the Minkowski metric for flat space:

$$\lim_{M\to 0} g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{1.3.28}$$

with the spatial part given in the usual spherical-polar coordinates, see equations (1.3.6) and (1.2.23).

By Birkhoff's theorem (see e.g. Appendix B of *Hawking & Ellis* [31]), if a space-time satisfying Einstein's equations contains an open subset U, which is source-free ($T_{\mu\nu} = 0$) and spherically symmetric, then U is isometric to an open subset of the Schwarzschild space-time.

Birkhoff's theorem also states that all spherically symmetric vacuum solutions to Einstein's equations must be static. This implies that the vacuum space-time outside any spherically-symmetric object is described by the Schwarzschild metric (1.3.27).

Singularities & Horizons

By inspection of (1.3.27), it can be seen that the Schwarzschild metric is "singular" at two values of the radial coordinate, $r = 2M$ and $r = 0$. The radius $r = 2M$ is known as the *Schwarzschild radius*, and can be seen to give a "coordinate singularity", which vanishes with an appropriate change of coordinates (e.g. *Eddington-Finkelstein* coordinates, see *Hawking & Ellis* [31], *Wald* [49]). However, the singularity at $r = 0$ is a physical singularity of the space-time itself.

When the Schwarzschild radius is approached form the outside, the outward directed null geodesics start tending towards the time-like Killing vector. *Inside* the Schwarzschild radius, all null geodesics end at the singularity $r = 0$, in a finite proper time. Thus the surface at $r = 2M$ is known as an "event horizon", separating the causal structure of the space-time, with no event at $r < 2M$ capable of influencing any event at $r > 2M$.

Due to the behavior of the null geodesics inside the event horizon, any spherically symmetric object with its mass contained entirely within the Schwarzschild radius, must undergo a "gravitational collapse" to a singular point-mass, creating what is known as a *black hole*. However, if the object exceeds the Schwarzschild radius, then the exterior vacuum solution described by the Schwarzschild metric, contains no singularities.

Schwarzschild Isotropic Coordinates

The Schwarzschild metric can also be given in coordinates that give the spatial part as a differential function times a flat 3-space metric. These coordinates are known as *isotropic* Schwarzschild coordinates, and are found by transforming the radial component by:

$$r \;=\; R\left(1+\frac{M}{2R}\right)^2 , \tag{1.3.29}$$

with R known as the *isotropic* radius. The line element in the isotropic form is found by applying the transform (1.3.29) to the metric (1.3.27), giving:

$$ds^2 \;=\; -\left(\frac{1-M/2R}{1+M/2R}\right)^2 dt^2 + \left(1+\frac{M}{2R}\right)^4 \left(dR^2 + R^2 d\theta^2 + R^2\sin^2\theta d\phi^2\right) . \tag{1.3.30}$$

Since the coordinates (R, θ, ϕ) are essentially flat, they can be transformed into cartesian type coordinates (x, y, z), giving the line element:

$$\begin{aligned} ds^2 \;&=\; -\left(\frac{1-M/2R}{1+M/2R}\right)^2 dt^2 + \left(1+\frac{M}{2R}\right)^4 \left(dx^2 + dy^2 + dz^2\right) , \\ R \;&=\; \sqrt{x^2+y^2+z^2} . \end{aligned} \tag{1.3.31}$$

This form of the metric proves to be particularly useful for many numerical simulations due to the lack of trigonometric functions.

In isotropic coordinates, the Schwarzschild radius at $R = M/2$ does not give a singular surface. However, both regions $R > M/2$ and $R < M/2$ can be seen from (1.3.29), to be given in Schwarzschild coordinates by the region $r > 2M$. The isotropic metric thus avoids the geometry inside the event horizon, by the "attachment" of a second *isometric* asymptotically flat region.

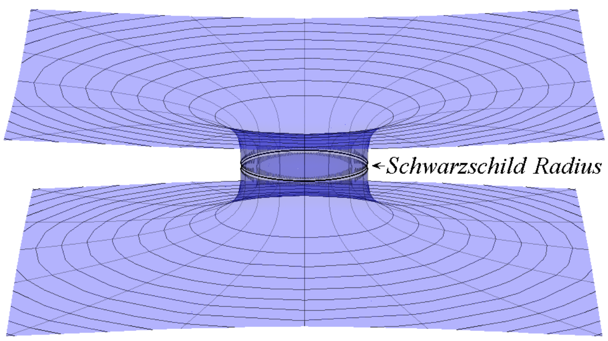


Figure 1.8: Representation of a time-slice of Schwarzschild space-time in isotropic coordinates, each coordinate circle representing a coordinate sphere.

The Schwarzschild solution, despite its simplicity, has proved extremely useful in the study of general relativity. Many astronomical objects approximate spherical symmetry, and asymptotic flatness when taken in isolation, including our solar system. As a result, the Schwarzschild solution has played a major role in the testing of the theory of general relativity, providing a remarkably accurate model for the precession of Mercury's orbit around the sun and the bending of starlight passing close to the sun.

The simplicity of the solution, particularly in isotropic form, makes it very useful in numerical relativity. Since the full metric is known exactly, it can be used as a test for numerical codes. The Schwarzschild solution has also been used extensively, to test analytically for the stability of numerical formulations.

1.3.4 Kerr Solution

Most astronomical objects have a non-negligible rotation, and hence cannot be assumed to be static. By Birkhoff's theorem, the Schwarzschild solution cannot hold for non-static space-times, so spherical-symmetry must give way to, at most, axial-symmetry. However, despite the Schwarzschild solution being found so quickly, a solution for an *axially*-symmetric, *stationary* space-time, took nearly fifty years to be discovered.

It was known from the weak-field approximations, that far from a single rotating source, the line element would take the approximate form:

$$\begin{aligned} ds^2 \; = \; & -\left(1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right) dt^2 - \left(\frac{4J\sin^2\theta}{r} + O\left(\frac{1}{r^2}\right)\right) dtd\phi \\ & + \left(1 + \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right)\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta \; d\phi^2\right) \; , \end{aligned} \tag{1.3.32}$$

in spherical-polar type coordinates, with M the mass of the source and J its angular momentum.

In 1963, an exact solution for a stationary, axially-symmetric space-time was finally found by Roy Kerr [33], giving the metric:

$$\begin{aligned} ds^2 \; = \; & (r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta \; d\phi^2) \\ & + 2(du + a \; \sin^2\theta \; d\phi)(dr + a \; \sin^s\theta \; d\phi) \\ & - \left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right)(du + a \; \sin^2\theta \; d\phi)^2 \; , \end{aligned} \tag{1.3.33}$$

for coordinates (u, r, θ, ϕ), with M representing the Schwarzschild mass, and Ma giving the angular momentum J.

Boyer-Lindquist Coordinates

In *Boyer-Lindquist* coordinates (see e.g. *Wald* [49]), a "time" coordinate t is taken to coincide with a time-like Killing vector, and a coordinate ϕ, with an axially-symmetric Killing vector, giving the Kerr metric, in coordinates (t, r, θ, ϕ):

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2Mr}{\Sigma}\right) & 0 & 0 & -\frac{2Mra \ \sin^2\theta}{\Sigma} \\ 0 & \frac{\Sigma}{\triangle} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\frac{2Mra\sin^2\theta}{\Sigma} & 0 & 0 & \left(\frac{(r^2+a^2)^2 - \triangle a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta \end{pmatrix} , \tag{1.3.34}$$

with the symbols $\triangle$ and Σ given by:

$$\begin{aligned} \triangle &= r^2 - 2Mr + a^2 \ , \\ \Sigma &= r^2 + a^2\cos^2\theta \ . \end{aligned} \tag{1.3.35}$$

As with Kerr's form of the metric (1.3.33), the Schwarzschild mass is given by M, and and the angular momentum J given by $J = Ma$, with "a" known as a *rotational* factor.

It can be seen that the Kerr metric approaches the form of the weak-field approximation (1.3.32), as the radial coordinate r becomes large. Taking $a = 0$ immediately gives the Schwarzschild metric (1.3.27), and it can be seen, though not easily, that for $M = 0$, (1.3.34) reduces to the flat Minkowski metric, in spherical coordinates (1.3.6).

Unlike the Schwarzschild solution, there is no equivalent Birkhoff theorem to imply that the Kerr solution is the unique stationary, axially symmetric, vacuum solution to Einstein's equations. However, it does seem likely that the Kerr space-time, containing the Schwarzschild space-time as a special case, provides the only solution for the "final" state of a charge-free black hole (see *Hawking & Ellis* [31], *Wald* [49]).

Singularities and Horizons

Singularities for the Kerr metric occur when Σ and $\triangle$ vanish. The first, given by:

$$r^2 + a^2\cos^2\theta \ = \ 0 \ , \tag{1.3.36}$$

is a physical singularity which, despite its appearance, takes the form of a ring in the equatorial plane, with diameter a. This reduces to the point singularity in the Schwarzschild limit.

The second singularity, given by $\triangle = 0$, is a coordinate singularity:

$$r^2 - 2Mr + a^2 \;=\; 0 \;,$$

$$\Leftrightarrow \qquad r_\pm \;=\; M \pm \sqrt{M^2 - a^2} \;, \tag{1.3.37}$$

giving two surfaces at r_+ and r_-, which both coincide with the Schwarzschild radius in the limit $a \to 0$. The outer of the two, r_+, is considered the event horizon, since null geodesics cannot pass through the surface in an outward direction.

Unlike the Schwarzschild space-time, the Kerr space-time contains a region outside the event horizon, where the time-symmetry Killing vector is no longer time-like. Hence, in this region, the t coordinate in the Boyer-Lindquist coordinates becomes *space*-like, changing sign. The boundary of this region, from the asymptotically flat region of the space-time, occurs where $g_{tt} = 0$:

$$\begin{aligned} 0 &= 1 - \frac{2Mr}{\Sigma} \\ &= r^2 + a^2 \cos^2\theta - 2Mr \;, \end{aligned}$$

$$\Leftrightarrow \qquad r \;= M + \sqrt{M^2 - a^2\cos^2\theta} \;, \tag{1.3.38}$$

which coincides *with* the event horizon (1.3.37) on the axis of rotation. This surface is known as the *ergosurface*, or the surface of static limit. The region between this surface and the event horizon is known as the ergosphere, within which no observer can remain static with respect to the space-time at infinity. This effect is also known as *frame dragging*, with inertial frames "dragged" in the direction of the mass's rotation.

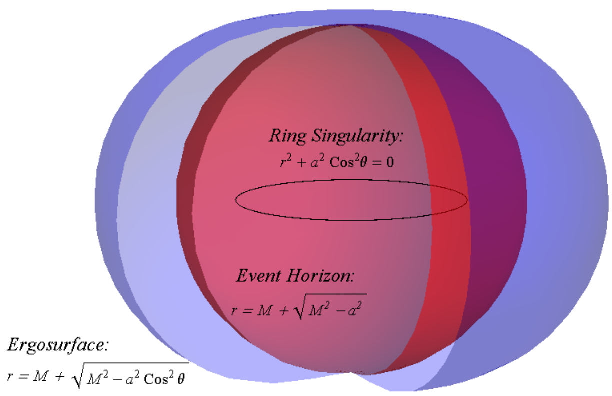


Figure 1.9: The physical "ring" singularity, event horizon, and surface of static limit for the Kerr space-time, with $a = 0.93M$.

Chapter 2

$3 + 1$ Numerical Relativity

2.1 The $3+1$ Formalism

Since so few exact solutions of Einstein's equations exist, other methods need to be found to make use of the theory of relativity in more complicated situations. The advent of computer technology and its ability to numerically solve integrals to high degrees of accuracy provides a useful tool for this.

To numerically solve Einstein's equations, for a particular set of conditions, the problem is best posed as a Cauchy initial value problem. The formalism that best deals with this is the $3+1$ formalism.

This formalism breaks the 4-dimensional space-time manifold back into 3-dimensional space-like surfaces, which fit together with a time parameter. Initial conditions are then set by providing a metric and its "time" derivative on an "initial" space-like surface. These are then integrating along the time parameter, using the second derivatives of the metric provided by Einstein's equations, to give the 3-metrics and their derivatives for subsequent space-like surfaces. The 4-metric can then be found from these 3-metrics and the time parameter, for at least an open subset of the space-time manifold.

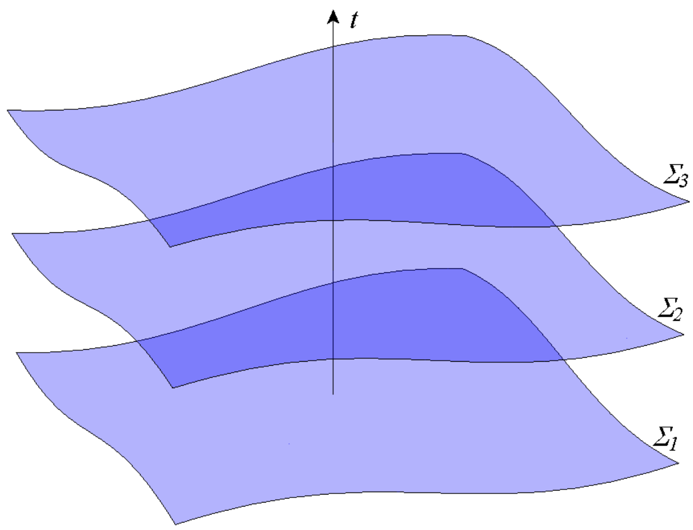


Figure 2.1: A foliation of space-like hypersurfaces (one spatial dimension suppressed).

The form given here was first put forward by *Arnowitt, Deser & Misner* [4] in 1962, with the use of the extrinsic curvature, and hence the form of the evolution equations, outlined in full by *York* [54] in 1979.

2.1.1 Space-time Foliations

For the $3 + 1$ formalism, the space-time manifold is assumed to contain a future pointing time-like covector field Ω_μ, given by the action of the covariant derivative operator, on a differential function t:

$$\Omega_\mu \ := \ \nabla_\mu t \ \equiv \ \partial_\mu t \qquad \text{s.t.} \qquad g^{\mu\nu}\Omega_\mu\Omega_\nu \ < \ 0 \ . \tag{2.1.1}$$

The commutator of two derivative operators acting on t gives:

$$\begin{aligned} \nabla_\mu\nabla_\nu t - \nabla_\nu\nabla_\mu t \ &= \nabla_\mu\partial_\nu t - \nabla_\nu\partial_\mu t \\ &= \partial_\mu\partial_\nu t - \partial_\nu\partial_\mu t - \cancel{\Gamma^\kappa_{\mu\nu}\partial_\kappa t} + \cancel{\Gamma^\kappa_{\nu\mu}\partial_\kappa t} \\ &= 0 \ , \end{aligned} \tag{2.1.2}$$

by the symmetry of the lower indices in the connection coefficients, and the equivalence of mixed partial derivatives for a differential function. Equations (2.1.1) and (2.1.2) thus lead to:

$$\nabla_\mu\Omega_\nu - \nabla_\nu\Omega_\mu \ = \ 0 \ , \tag{2.1.3}$$

satisfying equation (1.2.82). Hence, by Frobenius' theorem 2, the space-time manifold can be foliated into a family of 3-dimensional integral submanifolds.

These submanifolds are the level surfaces of the differential function t, and are therefore given by *space-like* hypersurfaces, and denoted by Σ_t, see figure 2.1. Since Ω_μ is defined to be time-like and future-pointing, t must be monotonically increasing in the future time-like direction, and therefore acts as a "time" parameter on the foliation. As a result, the space-like hypersurfaces Σ_t, are also known as *time slices* of the space-time manifold.

To avoid causality problems for the Cauchy problem, the $3+1$ formalism cannot be used for space-time manifolds that contain closed time loops. Hence, the covector field Ω_μ must be defined such that the integral curves of its associated vector fields Ω^μ, are nowhere closed on the space-time manifold.

The unit normal vector to the space-like hypersurfaces can be found from (2.1.1):

$$n^\mu \ := \ -\alpha \ g^{\mu\nu} \ \Omega_\nu \ , \tag{2.1.4}$$

with the normalizing term α, known as the "lapse" of normal time, found from the norm of Ω:

$$|\Omega|^2 \ = \ g^{\mu\nu} \ \nabla_\mu t \ \nabla_\nu t \ = \ -\frac{1}{\alpha^2} \ . \tag{2.1.5}$$

The negative sign in equation (2.1.4) ensures that the unit normal points in the direction of increasing t when the lapse α is positive. Also, since Ω_μ is defined to be time-like, the unit normal vector must also be time-like:

$$n^\mu n_\mu \ = \ g_{\mu\nu}\, n^\mu n^\nu \ = \ -1 \ , \tag{2.1.6}$$

and can therefore be considered as the 4-velocity of an *Eulerian* or *normal* observer.

The spatial metric $\gamma_{\mu\nu}$, induced by the space-time 4-metric $g_{\mu\nu}$, on each time slice, can now be given by:

$$\gamma_{\mu\nu} \ := \ g_{\mu\nu} + n_\mu\, n_\nu \ . \tag{2.1.7}$$

Since each time slice is an integral submanifold, and hence a manifold itself, the spatial metric acts as a metric in its own right on each Σ_t, according to section 1.2.2. The inverse of the spatial metric, is thus given by:

$$\gamma^{\mu\nu} \ \equiv \ g^{\mu\nu} + n^\mu n^\nu \ , \tag{2.1.8}$$

and with indices raised and lowered by $\gamma_{\mu\nu}$ and $\gamma^{\mu\nu}$ for any tensors defined purely on a given time slice. The spatial metric also operates as a projection operator onto the time slices, in the form:

$$\gamma^\mu_\nu \ = \ \delta^\mu_\nu + n^\mu n_\nu \ , \tag{2.1.9}$$

where the unit time-like normals cancel off the normal components of any vectors, covectors or tensors it acts on.

2.1.2 Coordinate Adaptations

Since the time slices Σ_t are integral submanifolds, there must exist a 3-dimensional span of vector fields, such that if an integral curve of one of these vector fields contains an element of a particular integral submanifold, then the curve is entirely contained in that submanifold. Hence, coordinates can be adapted to the foliation, so that three of the basis vector fields are elements of this span, and are therefore constrained to be tangent to the time slices Σ_t, at each point of the space-time manifold.

As a result of this adaptation, any tensor considered intrinsically on a time slice submanifold, can be determined entirely by the adapted spatial coordinates. To distinguish between space-time tensors, indices for tensors intrinsic to the time slice submanifolds are given by *Latin* indices from here on, ranging over the *three* spatial coordinates alone. For example, considered on a given time slice Σ_t, ignoring the space-time embedding of Σ_t, the spatial metric is denoted by:

$$\gamma_{\mu\nu}\big|_{\Sigma_t} \ := \ \gamma_{ab} \ , \tag{2.1.10}$$

with a and b ranging over any three coordinates defined on the time slice.

A "time" vector field t^μ can now be chosen to relate the spatial coordinates, between two infinitesimally close time slices. This vector field does *not* have to be time-like, since there is nothing to stop *coordinates* changing faster than light, but it must not coincide with any time slice tangent vectors at any point. The time vector is therefore defined such that:

$$\begin{aligned} t^\mu \Omega_\mu &:= 1 \\ \Rightarrow \qquad t^\mu &= \alpha n^\mu + \beta^\mu \ , \end{aligned} \tag{2.1.11}$$

where β^μ is a purely spatial vector field, giving the "shift" of spatial coordinates in time, with respect to the normal observer, and is generally known as the *shift* vector.

It is now natural to adapt the coordinates further, by choosing the time coordinate to coincide with the time vector t^μ:

$$t^\mu \ := \ (1,\ 0,\ 0,\ 0) \ . \tag{2.1.12}$$

Since the shift vector field is entirely spatial, in space-time coordinates $\beta^\mu = (0,\ \beta^i)$, and hence the unit normal vector and covector are given by:

$$n^\mu \ = \ \frac{1}{\alpha}\left(1,\ -\beta^i\right) \ , \qquad n_\mu \ = \ -\left(\alpha,\ 0,\ 0,\ 0\right) \ . \tag{2.1.13}$$

The spatial metric, in this coordinate system, can easily be shown from equations (2.1.7) and (2.1.13), to be equivalent to the spatial part of the space-time metric, $\gamma_{ij} \equiv g_{ij}$.

The space-time metric, adapted to this coordinate system, is thus given by:

$$g_{\mu\nu} \ = \ \begin{pmatrix} -\alpha^2 + \beta_s \beta^s & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix} \ , \qquad g^{\mu\nu} \ = \ \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix} \ , \tag{2.1.14}$$

from which the space-time line element is given by:

$$ds^2 \ = \ -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right) \ . \tag{2.1.15}$$

Equations (2.1.14) and (2.1.15) are considered to represent the space-time metric "decomposed" into the 3 + 1 formalism.

The lapse and shift, together, form a set of 4 *gauge* conditions in the 3+1 formalism. They represent 4 degrees of freedom, related to the "rate of change" of coordinates along the unit time-like normal n^μ. The lapse, representing the advancement of the time coordinate with respect to proper time, can be chosen differently at each point on a space-like hypersurface, to take advantage of different properties of a space-time manifold. The shift vector can also be chosen arbitrarily, allowing spatial coordinates to be reassigned as needed between slices.

2.1.3 Extrinsic Curvature

The extrinsic curvature of a surface in a manifold, is a measure of how that surface is curved within the manifold. In differential geometry, the extrinsic curvature is generally known as the *second fundamental form*, with the *first* fundamental form given by the metric. On a space-like hypersurface, the curvature is given by *projecting* a parallel transported unit normal vector, *onto* the hypersurface.

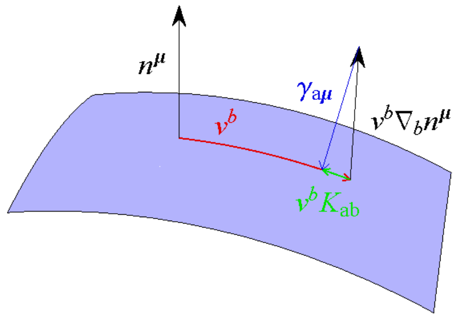


Figure 2.2: The extrinsic curvature $v^b K_{ab}$, along a spatial vector field v^b, as the projection of the normal vector n^μ, parallel transported along v^b.

To ensure that the extrinsic curvature tensor is symmetric about its indices, however, it is defined by:

$$K_{ab} := \gamma^\mu_a \; \gamma^\nu_b \; \frac{1}{2} \left(\nabla_\mu n_\nu + \nabla_\nu n_\mu \right) \;, \tag{2.1.16}$$

and can be seen by the projection operators, to be defined entirely within each integral submanifold. It can also be seen, from equation (1.2.53), that the terms inside the brackets in (2.1.16), are equivalent to the Lie derivative of the space-time metric $g_{\mu\nu}$, with respect to the unit normal vector n^μ. Substituting into equation (2.1.16), the projection operators can be seen to give:

$$\begin{aligned} K_{ab} &= \gamma^\mu_a \; \gamma^\nu_b \; \frac{1}{2} \; \mathcal{L}_n g_{\mu\nu} \\ &= \frac{1}{2} \; \mathcal{L}_n \gamma_{ab} \;, \end{aligned} \tag{2.1.17}$$

showing the extrinsic curvature to be equivalent to the rate of change of the spatial metric, as it is "Lie dragged" normal to the time slice.

Since (2.1.17) shows the extrinsic curvature to give a time-like derivative of the spatial metric, the Lie derivative, with respect to the *time* vector t^μ, of the spatial metric gives:

$$\mathcal{L}_t \gamma_{ab} = \alpha \; \mathcal{L}_n \gamma_{ab} + \mathcal{L}_\beta \gamma_{ab} \;, \tag{2.1.18}$$

due to the linearity of the Lie derivative. With t^μ coinciding with the "time" coordinate, equation (2.1.12), the Lie derivative with respect to t^μ is reduced to:

$$\mathcal{L}_{t^\mu} \equiv \partial_t \;, \tag{2.1.19}$$

similar to the Lie derivative of a vector field along a coordinate vector in (1.2.45).

Also, since the shift vector is purely spatial, the Lie derivative with respect to the shift vector can be expanded according to equation (1.2.53), reducing (2.1.18) to:

$$\partial_t \gamma_{ab} \;=\; 2\;\alpha K_{ab} + D_a \beta_b + D_b \beta_a \;, \tag{2.1.20}$$

where $D_a := \gamma_a^\mu \nabla_\mu$ is the projection of the covariant derivative onto the time slices Σ_t, giving the covariant derivative associated with the spatial metric γ_{ab} on each time slice.

Since equation (2.1.20) gives the time evolution of the spatial metric, the "initial conditions" can be given by simply defining the spatial metric γ_{ab} and extrinsic curvature K_{ab} on an "initial" space-like hypersurface (see *Smarr & York* [47], *York* [54]).

The mean curvature, denoted by K, is given by the trace of the extrinsic curvature:

$$K \;:=\; \gamma^{ab} K_{ab} \;. \tag{2.1.21}$$

Taking the trace of equation (2.1.17) for K_{ab}, the mean curvature can be given by:

$$K \;=\; \mathcal{L}_n \ln \gamma^{1/2} \;, \tag{2.1.22}$$

showing K to represent the fractional change of the 3-space volume element, along the normal vector n^μ.

2.1.4 Einstein Constraint Equations

With a space-time manifold foliated into space-like integral submanifolds, and coordinates adapted according to section 2.1.2, these submanifolds are still required to satisfy Einstein's field equations. Since Einstein's equations (1.3.17) depend on the 4-space Ricci curvature, which is found from the Riemann tensor, then if conditions are to be given intrinsic to the time slices Σ_t, the 4-Riemann tensor must be decomposed into the $3+1$ formalism.

The equations of Gauss and Codazzi give projections of the 4-Riemann tensor into submanifolds of a manifold, in terms intrinsic to the submanifold (see e.g. *Kobayashi & Nomizu* [36] (chapter VII, section 4) for derivations):

$$\gamma_a^\mu\;\gamma_b^\nu\;\gamma_c^\rho\;\gamma_d^\sigma\;{}^{(4)}R_{\mu\nu\rho\sigma} \;=\; {}^{(3)}R_{abcd} + K_{ac}K_{bd} - K_{ad}K_{cb} \;, \tag{2.1.23}$$

$$\gamma_a^\mu\;\gamma_b^\nu\;\gamma_c^\rho\;n^\sigma\;{}^{(4)}R_{\mu\nu\rho\sigma} \;=\; -\,D_b K_{ac} + D_a K_{bc} \;. \tag{2.1.24}$$

Combining these equations with the Einstein field equations then give conditions on the choice of spatial metric and extrinsic curvature on each time slice.

Taking Gauss' equation (2.1.23), and contracting twice to get the 4-Ricci tensor:

$$\begin{aligned}
\gamma^{ac}\gamma^{bd}\ \gamma^{\mu}_{a}\gamma^{\nu}_{b}\gamma^{\rho}_{c}\gamma^{\sigma}_{d}\ {}^{(4)}R_{\mu\nu\rho\sigma} &= \gamma^{ac}\gamma^{bd}\left({}^{(3)}R_{abcd} + K_{ac}K_{bda} - K_{ad}K_{cb}\right)\ , \\
\Leftrightarrow \qquad {}^{(3)}R + K^2 - K_{ab}\ K^{ab} &= \gamma^{\mu\rho}\gamma^{\nu\sigma}\ {}^{(4)}R_{\mu\nu\rho\sigma} \\
&= {}^{(4)}R + 2n^{\mu}n^{\nu}\ {}^{(4)}R_{\mu\nu} \\
&= 16\ \pi\ n^{\mu}n^{\nu}T_{\mu\nu}\ ,
\end{aligned} \tag{2.1.25}$$

substituting the stress-energy tensor from the field equations (1.3.17). Defining the *energy density* as the total energy measured by a normal observer:

$$\rho\ :=\ n^{\mu}n^{\nu}T_{\mu\nu}\ , \tag{2.1.26}$$

and substituting this into equation (2.1.25), gives the *Hamiltonian* constraint:

$${}^{(3)}R + K^2 - K_{ab}\ K^{ab}\ =\ 16\pi\rho\ . \tag{2.1.27}$$

Taking now the Codazzi equation (2.1.24), and contracting once this time:

$$\begin{aligned}
\gamma^{bc}\ \gamma^{\mu}_{a}\gamma^{\nu}_{b}\gamma^{\rho}_{c}n^{\sigma}\ {}^{(4)}R_{\mu\nu\rho\sigma} &= \gamma^{bc}\left(-D_bK_{ac} + D_aK_{bc}\right)\ , \\
\Leftrightarrow \qquad D_bK_{ab} - D_aK &= -\ \gamma^{\mu}_{a}\gamma^{\nu\rho}n^{\sigma}\ {}^{(4)}R_{\mu\nu\rho\sigma} \\
&= \gamma^{\mu}_{a}n^{\sigma}\ {}^{(4)}R_{\mu\sigma} + \gamma^{\mu}_{a}n^{\nu}n^{\rho}n^{\sigma}\ {}^{(4)}\cancelto{0}{R_{\nu\mu\rho\sigma}} \\
&= \gamma^{\mu}_{a}n^{\sigma}\left({}^{(4)}R_{\mu\sigma} + \frac{1}{2}\ g_{\mu\sigma}\ {}^{(4)}R\right) \\
&= 8\pi\gamma^{\mu}_{a}n^{\sigma}T_{\mu\sigma}\ ,
\end{aligned} \tag{2.1.28}$$

substituting the stress-energy tensor again from Einstein's equations (1.3.17). Defining now, the *momentum* density measured by a normal observer:

$$j^{a}\ :=\ \gamma^{\mu a}n^{\nu}T_{\mu\nu}\ , \tag{2.1.29}$$

and raising the free index in equation (2.1.28), gives the *momentum* constraint:

$$D_b\left(K^{ab} - \gamma^{ab}K\right)\ =\ -8\pi j^{a}\ . \tag{2.1.30}$$

Both equations (2.1.27) and (2.1.30) lie completely in the space-like hypersurfaces, and relate the spatial metric and extrinsic curvature to projections of the stress-energy tensor. As a result, they are considered as "constraint" equations on the space-like hypersurfaces, and in particular the "initial" hypersurface, by restricting the choices of γ_{ab} and K_{ab}, to those that satisfy Einstein's equations.

2.1.5 Evolution Equations

The constraint equations (2.1.27) and (2.1.30) give 4 of the 10 Einstein equations, in the $3+1$ formalism. The other 6 equations require the remaining terms from the decomposition of the 4-Riemann tensor. These terms can be seen to be given by the equation:

$$\gamma_a^\sigma \gamma_b^\nu n^\mu n^\rho \, {}^{(4)}R_{\mu\nu\rho\sigma} \;=\; -\mathcal{L}_n K_{ab} + \frac{1}{\alpha} D_a D_b \alpha + K_b^c K_{ac} \;, \tag{2.1.31}$$

from *York* [54] (equation (32)).

With the evolution of the spatial metric given by the extrinsic curvature in equation (2.1.20):

$$\partial_t \gamma_{ab} \;=\; 2\,\alpha K_{ab} + D_a \beta_b + D_b \beta_a \;, \tag{2.1.32}$$

it remains then, to find an evolution equation for the curvature K_{ab} itself. Since Einstein's equations are only second order in the space-time metric, no higher order equations are necessary.

As with the evolution of the spatial metric, the evolution of the extrinsic curvature with respect to the adapted time coordinate of equation (2.1.12), is given from equation (2.1.19) by its Lie derivative with respect to the time vector t^μ:

$$\partial_t K_{ab} \;=\; \mathcal{L}_t K_{ab} \;=\; \alpha \, \mathcal{L}_n K_{ab} + \mathcal{L}_\beta K_{ab} \;, \tag{2.1.33}$$

again using the linearity of the Lie derivative, and the definition of the time vector (2.1.11). The Lie derivative with respect to the shift vector can be expanded similar to (2.1.20), and equation (2.1.31) can be rearranged to give an expression for $\mathcal{L}_n K_{ab}$:

$$\gamma_a^\sigma \gamma_b^\nu n^\mu n^\rho \, {}^{(4)}R_{\mu\nu\rho\sigma} \;=\; -\mathcal{L}_n K_{ab} + \frac{1}{\alpha} D_a D_b \alpha + K_b^c K_{ac} \;,$$

$$\Leftrightarrow \qquad \mathcal{L}_n K_{ab} \;=\; -\,\gamma_a^\sigma \gamma_b^\nu n^\mu n^\rho \, {}^{(4)}R_{\mu\nu\rho\sigma} + \frac{1}{\alpha} D_a D_b \alpha + K_b^c K_{ac} \;, \tag{2.1.34}$$

though in terms of a projection of the 4-Riemann curvature: $\gamma_a^\sigma \gamma_b^\nu n^\mu n^\rho \, {}^{(4)}R_{\mu\nu\rho\sigma}$.

In order to give the evolution equation for the extrinsic curvature as part of a Cauchy formalism, the projection of ${}^{(4)}R_{\mu\nu\rho\sigma}$, appearing in (2.1.34), must be given in terms intrinsic to the time slices. The projection is therefore rewritten as:

$$\gamma_a^\sigma \gamma_b^\nu n^\mu n^\rho \, {}^{(4)}R_{\mu\nu\rho\sigma} \;=\; \gamma_a^\sigma \gamma_b^\nu \gamma^{\mu\rho} \, {}^{(4)}R_{\mu\nu\rho\sigma} - \gamma_a^\sigma \gamma_b^\nu \, {}^{(4)}R_{\nu\sigma} \;, \tag{2.1.35}$$

using equation (2.1.8), which relates the raised form of the spatial and space-time metrics.

The first term on the right hand side of equation (2.1.35) can be given by contracting the Gauss equation (2.1.23) once:

$$\gamma^{\mu\rho}\gamma^{\sigma}_{a}\gamma^{\nu}_{b}\ {}^{(4)}R_{\mu\nu\rho\sigma} = {}^{(3)}R_{ab} + KK_{ab} - K_{ac}K^{c}_{b}\ . \tag{2.1.36}$$

The second term can then be given by Einstein's field equations, in the form (1.3.19):

$${}^{(4)}R_{\nu\sigma} \ = \ 8\pi\left(T_{\nu\sigma} - \frac{1}{2}\ T\ g_{\nu\sigma}\right)\ . \tag{2.1.37}$$

Finally, it is necessary to give the spatial part of the stress-energy tensor, the final part of its $3+1$ decomposition, giving also its trace:

$$S_{ab} \ := \ \gamma^{\mu}_{a}\ \gamma^{\nu}_{b}\ T_{\mu\nu}\ , \qquad S \ := \ \gamma^{ab}S_{ab}\ . \tag{2.1.38}$$

Equation (2.1.35) can now be rewritten, using equations (2.1.36), (2.1.37) and (2.1.38):

$$\begin{aligned}
&\gamma^{\sigma}_{a}\gamma^{\nu}_{b}n^{\mu}n^{\rho}\ {}^{(4)}R_{\mu\nu\rho\sigma}\\
&\qquad = {}^{(3)}R_{ab} + KK_{ab} - K_{ac}K^{c}_{b} - \gamma^{\sigma}_{a}\gamma^{\nu}_{b}\ 8\ \pi\left(T_{\nu\sigma} - \frac{1}{2}\ T\ g_{\nu\sigma}\right)\\
&\qquad = {}^{(3)}R_{ab} + KK_{ab} - K_{ac}K^{c}_{b} - 8\ \pi\left(S_{ab} - \frac{1}{2}\ \gamma_{ab}(S-\rho)\right)\ ,
\end{aligned} \tag{2.1.39}$$

giving the projection of ${}^{(4)}R_{\mu\nu\rho\sigma}$, in terms intrinsic to the spatial hypersurface alone.

Substituting ${}^{(4)}R_{\mu\nu\rho\sigma}$ from (2.1.39) into (2.1.34) gives an expression for $\mathcal{L}_n K_{ab}$, purely in terms intrinsic to the spatial hypersurface:

$$\begin{aligned}
\mathcal{L}_n K_{ab} \ &= \ -\gamma^{\sigma}_{a}\gamma^{\nu}_{b}n^{\mu}n^{\rho}\ {}^{(4)}R_{\mu\nu\rho\sigma} + \frac{1}{\alpha}D_a D_b\alpha + K^{c}_{b}K_{ac}\\
&= \ -^{(3)}\ R_{ab} - KK_{ab} + K_{ac}K^{c}_{b}\\
&\qquad + 8\pi\left(S_{ab} - \frac{1}{2}\ \gamma_{ab}(S-\rho)\right) + \frac{1}{\alpha}D_a D_b\alpha + K^{c}_{b}K_{ac}\\
&= \frac{1}{\alpha}D_a D_b\alpha -^{(3)} R_{ab} - KK_{ab} + 2\ K_{ac}K^{c}_{b} + 8\pi\left(S_{ab} - \frac{1}{2}\ \gamma_{ab}(S-\rho)\right).
\end{aligned} \tag{2.1.40}$$

Adding the Lie derivative with respect to the shift vector to (2.1.40), gives equation (2.1.33) as the full evolution equation for the extrinsic curvature, as a time slice entity:

$$\begin{aligned}
\partial_t K_{ab} \ &= D_a D_b\alpha - \alpha\left({}^{(3)}R_{ab} + KK_{ab} - 2\ K_{ac}K^{c}_{b}\right)\\
&\qquad + 8\pi\alpha\left(S_{ab} - \frac{1}{2}\ \gamma_{ab}(S-\rho)\right) + \mathcal{L}_\beta K_{ab}\ .
\end{aligned} \tag{2.1.41}$$

The Lie derivative with respect to the shift vector can also be expanded, as with the evolution equation for the spatial metric (2.1.20), in a manner similar to (1.2.52).

2.1.6 The Cauchy Problem

The technique for solving Einstein's equations *using* the $3+1$ formalism takes a slightly different form than the derivation of the formalism.

To begin with, a set of initial conditions (γ_{ab}, K_{ab}) are defined on a space-like hypersurface Σ_0, such that the Hamiltonian and momentum constraints (2.1.27), (2.1.30) are satisfied:

$$R + K^2 - K_{ab}\,K^{ab} \;=\; 16\pi\rho\;, \tag{2.1.42}$$

$$D_b\left(K^{ab} - \gamma^{ab}K\right) \;=\; -\,8\pi j^a\;, \tag{2.1.43}$$

with the source terms given by the decomposition of the stress-energy tensor:

$$\rho \;=\; n^\mu n^\nu T_{\mu\nu}\;, \qquad j^a \;=\; \gamma^{\mu a} n^\nu T_{\mu\nu}\;, \qquad S_{ab} \;=\; \gamma^\mu_a\,\gamma^\nu_b\,T_{\mu\nu}\;, \tag{2.1.44}$$

noting that, from here on, the "(3)" superscript is dropped from the spatial curvatures. Unfortunately, finding initial conditions that satisfy the constraint equations is less than trivial. Techniques have been developed for doing so however, with the more successful ones given in section 2.2.

Once initial conditions have been found, the gauge conditions of the lapse α and shift β^a need to be chosen, to give the time vector t^μ from equation (2.1.11), and hence the time coordinate t. Again, good choices for the gauge conditions are far from obvious, with some of the more common conditions outlined in section 2.3.

With initial and gauge conditions chosen, the spatial metric and extrinsic curvature can be evolved along the time vector t^μ, by numerical integration of the evolution equations (2.1.20) and (2.1.41):

$$\partial_t\,\gamma_{ab} \;= 2\,\alpha K_{ab} + D_a\beta_b + D_b\beta_a\;, \tag{2.1.45}$$

$$\begin{aligned}\partial_t\,K_{ab} \;&= D_aD_b\alpha + \beta^c D_cK_{ab} + K_{ac}D_b\beta^c + K_{cb}D_a\beta^c \\ &\quad - \alpha\left(R_{ab} + KK_{ab} - 2\,K_{ac}K^c_b\right) + 8\pi\alpha\left(S_{ab} - \frac{1}{2}\,\gamma_{ab}(S-\rho)\right)\;,\end{aligned} \tag{2.1.46}$$

for each time step, giving a set of spacelike hypersurface Σ_t.

Finally, the spatial metric, lapse and shift, are substituted into equation (2.1.14):

$$g_{\mu\nu} \;=\; \begin{pmatrix} -\alpha^2 + \beta_s\beta^s & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}\;, \qquad g^{\mu\nu} \;=\; \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix}\;, \tag{2.1.47}$$

giving the space-time 4-metric $g_{\mu\nu}$, at each "evaluated" point of the evolution.

2.1.7 Exact Solutions in the $3 + 1$ Formalism

The exact solutions of Schwarzschild and Kerr, see sections 1.3.3 and 1.3.4, can be given in the 3+1 formalism by equating their space-time metrics with equation (2.1.14). The lapse, shift and spatial metric can then be found for the coordinate system of the given metric.

For the Schwarzschild metric, with the time coordinate coinciding with a time-like Killing vector, the shift vector can be seen to be zero, see equation (1.3.20). The lapse is then given, by:

$$\alpha = \sqrt{1 - \frac{2M}{r}} \,, \qquad \alpha = \frac{1 - M/2R}{1 + M/2R} \,, \tag{2.1.48}$$

in ordinary Schwarzschild and in isotropic coordinates respectively, from equations (1.3.27) and (1.3.30). The same can be done for the Kerr metric, though with a non-zero shift vector, the calculations become more complicated for the lapse.

The spatial metrics for the Schwarzschild space-time, in ordinary and cartesian isotropic coordinates respectively, are then given by:

$$\gamma_{ab} = \begin{pmatrix} \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \,, \tag{2.1.49}$$

$$\gamma_{ab} = \left(1 + \frac{M}{2R}\right)^4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,, \tag{2.1.50}$$

from equations (1.3.27) and (1.3.31). The extrinsic curvature can then be calculated directly from equation (2.1.17):

$$K_{ab} = \frac{1}{2} \mathcal{L}_n \gamma_{ab} \,. \tag{2.1.51}$$

As mentioned in section 1.3, the Schwarzschild and Kerr solutions, decomposed into the $3 + 1$ formalism, can be used in the testing of numerical codes, since the solutions are given everywhere for comparison with.

2.2 Initial Conditions

The initial conditions for evaluation in the $3+1$ formalism are given by the spatial metric and extrinsic curvature on an "initial" space-like hypersurface. Since both tensors are completely spatial and symmetric, the pair $(\gamma_{ab},\ K_{ab})$ contains 12 independent components, once a spatial coordinate system has been chosen.

Due to the Hamiltonian and momentum constraints, these 12 components cannot be chosen arbitrarily. The constraint equations (2.1.42), (2.1.43), determine 4 of these components, once the other 8 have been chosen. However, there is no obvious way to distinguish which of the 12 components are freely specifiable, and which 4 should be constrained.

The most successful methods for forming initial conditions have involved conformal re-scaling of the metric, and transverse trace-free decomposition of the extrinsic curvature. These techniques separate particular properties of the metric and curvature, so that the choice of which to specify freely, and which to constrain, can be made more effectively.

The conformal transverse trace-free decomposition uses both these techniques, and has been one of the most successful methods of determining initial conditions. The conformal thin sandwich decomposition then combines these techniques with the gauge choices for the lapse and shift, giving a set of freely specifiable data that completely determines the evolution of the system.

2.2.1 Conformal Transformations of the Spatial Metric

Conformal transformation of a tensor involves taking the scaling factor of the tensor as an independent variable. This reduces the number of independent components in the tensor by one. For the spatial metric this separates one of the 6 independent variables from the metric.

For a positive scaling factor ψ, known as the conformal factor, the spatial metric is conformally transformed by:

$$\gamma_{ab} \;=\; \psi^4\ \bar{\gamma}_{ab}\ , \tag{2.2.1}$$

with $\bar{\gamma}_{ab}$ known as the conformal metric, or background metric. The power of 4 for the conformal factor is arbitrary, but turns out to be convenient.

Note from equation (2.1.50), that the Schwarzschild metric, in isotropic coordinates, gives a conformally *flat* spatial metric, with the conformal factor given by:

$$\psi \;=\; 1 + \frac{M}{2R} \;, \tag{2.2.2}$$

and the conformal metric given by the flat 3-space metric of Euclidean space. Thus, working in conformal terms of the Schwarzschild isotropic metric, all of the curvature tensors vanish, and the derivatives reduce to their Euclidean equivalents.

It is often useful to choose the determinant of the conformal metric to be 1, which associates the conformal factor with the determinant of the spatial metric:

$$\bar{\gamma} \;=\; 1 \qquad \Leftrightarrow \qquad \psi \;=\; \gamma^{1/12} \;, \tag{2.2.3}$$

with γ representing the determinant of the spatial metric γ_{ab}, and $\bar{\gamma}$ representing the determinant of the conformal metric $\bar{\gamma}_{ab}$. The conformal metric resulting from this transformation is referred to as the “natural” conformal metric, and is sometimes denoted specifically by a tilde: $\tilde{\gamma}_{ab} := \gamma^{-1/3}\gamma_{ab}$.

Since the metric is used to compute the connection coefficients, and the Ricci and Scalar curvatures through the Riemannian curvature tensor, these can now be represented in terms of the conformal metric $\bar{\gamma}_{ab}$ and the conformal factor ψ. Substituting equation (2.2.1) into (1.2.36) for the connection coefficients, and into (1.2.70) and then (1.2.71) and (1.2.72) for the curvatures, gives:

$$\Gamma^c_{ab} \;= \bar{\Gamma}^c_{ab} + 2\left(\delta^c_a\;\bar{D}_b \ln\psi + \delta^c_b\;\bar{D}_a \ln\psi - \bar{\gamma}_{ab}\;\bar{\gamma}^{cd}\;\bar{D}_d \ln\psi\right) \;, \tag{2.2.4}$$

$$\begin{aligned} R_{ab} \;&= \bar{R}_{ab} - 2\left(\bar{D}_a\bar{D}_b \ln\psi + \bar{\gamma}_{ab}\;\bar{\gamma}^{cd}\;\bar{D}_c\bar{D}_d \ln\psi\right) \\ &\quad + 4\left(\left(\bar{D}_a \ln\psi\right)\left(\bar{D}_b \ln\psi\right) - \bar{\gamma}_{ab}\;\bar{\gamma}^{cd}\left(\bar{D}_c \ln\psi\right)\left(\bar{D}_d \ln\psi\right)\right) \;, \end{aligned} \tag{2.2.5}$$

$$R \;= \psi^{-4}\bar{R} - 8\;\psi^{-5}\bar{D}^2\psi \;=\; \psi^{-4}\bar{R} - 8\;\psi^{-5}\;\bar{\gamma}^{ab}\;\bar{D}_a\bar{D}_b\;\psi \;, \tag{2.2.6}$$

with $\bar{D}_a$ representing the 3-covariant derivative associated with the conformal metric, i.e. $\bar{D}_a\bar{\gamma}_{ab} = 0$. All other objects associated with the conformal metric are also represented by a bar.

The Hamiltonian constraint (2.1.42) can now be expressed in conformal terms:

$$8\;\bar{D}^2\psi - \psi\;\bar{R} - \psi^5 K^2 + \psi^5 K_{ab}K^{ab} \;=\; -16\;\pi\;\psi^5\;\rho \;. \tag{2.2.7}$$

with the conformal factor now acting as one of the constrained variables, found by solving the conformal Hamiltonian equation above.

2.2.2 Transverse and Trace-Free Decomposition of Extrinsic Curvature

Similar to the conformal transformation of the spatial metric, decomposing the extrinsic curvature divides its 6 degrees of freedom between different tensors. This allows different parts of the extrinsic curvature to be used as free and constrained data. Most of the work on applying transverse and longitudinal decomposition, and trace and trace-free decomposition to the initial conditions of the $3+1$ formalism was first done by James W. York Jr., see e.g. *York* [53, 51], and *Ó Murchadha & York* [42].

It can be shown that any symmetric tensor can be split into two parts, one of which has a vanishing divergence. Since the extrinsic curvature is symmetric by definition, (2.1.16), it can thus be given by:

$$K^{ab} \;=\; K_T^{ab} + K_L^{ab} \;, \qquad \text{s.t.} \qquad D_b\, K_T^{ab} \;=\; 0 \;. \tag{2.2.8}$$

where T denotes the *transverse* (divergence-free) part and L, the *longitudinal* part. It can be shown that the longitudinal part K_L^{ab}, can be given by the equation:

$$K_L^{ab} \;=\; (KW)^{ab} := \; D^aW^b + D^bW^a \;, \tag{2.2.9}$$

for some vector W^a. The term $(KW)^{ab}$ is known as the *Killing form* with respect to W^a, since $(KW)^{ab} \equiv \mathcal{L}_W \gamma^{ab}$, see (1.2.74).

Before proceeding with the transverse splitting, however, the extrinsic curvature is first split into its trace and trace free parts:

$$K^{ab} \;=\; A^{ab} + \frac{1}{3}\gamma^{ab} K \;, \tag{2.2.10}$$

recalling K as the trace of K^{ab}, (2.1.21).

The trace of the curvature clearly removes one of the component freedoms from the trace-free part A^{ab}. Once 5 components are given for A^{ab}, in any particular coordinate system, the final component must be chosen to make the sum $\gamma_{ab}A^{ab}$ vanish, i.e. make A^{ab} "trace-free". Hence A^{ab} contains 5 degrees of freedom, with the remaining 1 given by K.

Since the extrinsic curvature is symmetric, the trace free part A^{ab} must also be symmetric, and can thus also be decomposed into transverse and longitudinal parts:

$$A^{ab} \;=\; A_{TT}^{ab} + A_L^{ab} \;, \qquad \text{s.t.} \qquad D_b\, A_{TT}^{ab} \;=\; 0 \;. \tag{2.2.11}$$

where the subscript TT indicates that A_{TT}^{ab} is both transverse and trace-free.

The longitudinal part A_L^{ab} now satisfies the equation:

$$A_L^{ab} \;=\; (LW)^{ab} \;:=\; D^a W^b + D^b W^a - \frac{2}{3}\,\gamma^{ab} D_k W^k \;, \tag{2.2.12}$$

for some vector W^a. The last term in the equation above comes from the removal of the trace influence from the longitudinal part of the full extrinsic curvature (2.2.9):

$$A_L^{ab} \;=\; K_L^{ab} \;-\; \left(\frac{1}{3}\gamma^{ab} K\right)_L \;. \tag{2.2.13}$$

This term can also be given by the Lie derivative of a tensor *density*, see (1.2.65):

$$(LW)^{ab} \;\equiv\; \gamma^{1/3} \mathcal{L}_W \left(\gamma^{-1/3}\gamma^{ab}\right) \;, \tag{2.2.14}$$

and since $\gamma^{-1/3}\gamma^{ab} = \tilde{\gamma}^{ab}$, which is the "natural" conformal metric (2.2.3), $(LW)^{ab}$ is referred to as the *conformal* Killing form, with respect to the vector W^a.

Note that if a spatial metric γ_{ab} contains a *conformal* Killing vector W^a, i.e.:

$$W^a \qquad \text{s.t.} \qquad (LW)^{ab} \;=\; 0 \;, \tag{2.2.15}$$

then any conformally related metric $\bar{\gamma}_{ab}$ also contains W^a as a conformal Killing vector, since the *natural* conformal metric $\tilde{\gamma}_{ab}$ from (2.2.3), removes any scale variance from any other conformal transformations of γ_{ab}, leaving $(LW)^{ab}$ conformally invariant.

The divergence of the trace-free part of the extrinsic curvature A^{ab} is then given, by construction, by the divergence of equation (2.2.12):

$$\begin{aligned} D_b A^{ab} &\equiv D_b \,(LW)^{ab} \\ &= D^2 W^a + \frac{1}{3} D^a \left(D_b W^b\right) + R^a_b W^b \\ &:= \left(\triangle_L W\right)^a \;, \end{aligned} \tag{2.2.16}$$

where $\triangle_L$ is a vector Laplacian, and can be shown to be an elliptic operator.

Note that 3 of the 5 component variables for the trace-free part of the extrinsic curvature, are given by the vector W^a in the longitudinal part, (2.2.12). In a sense, the vanishing of the 3 components of the divergence of A_{TT}^{ab}, see equation (2.2.11), acts as a set of constraint equations, reducing the component freedom of A_{TT}^{ab} by 3. Hence, the 6 component freedoms of the extrinsic curvature K^{ab} are assigned by:

$$K : 1 \text{ free component,} \tag{2.2.17a}$$

$$K^{ab} : \; 6 \text{ free components,} \qquad \rightarrow \qquad A_{TT}^{ab} : 2 \text{ free components,} \tag{2.2.17b}$$

$$A_L^{ab} : 3 \text{ free components.} \tag{2.2.17c}$$

2.2.3 Conformal Transverse Trace-free Decomposition

The conformal transverse trace-free decomposition of York and Lichnerowicz [39, 51, 52] is one of the most widespread methods used to form initial conditions in numerical relativity.

To begin with, the spatial metric is conformally transformed, according to equation (2.2.1). The extrinsic curvature is then conformally transformed, using the same conformal factor ψ (though not necessarily of the same power), in order to relate the curvature to the conformal metric. The conformally transformed extrinsic curvature is then decomposed according (2.2.17), and the momentum constraints adjusted to constrain the *longitudinal* part.

In conformally transforming the extrinsic curvature, it proves useful to transform the trace and trace-free parts separately:

$$A^{ab} \;=\; \psi^{p}\bar{A}^{ab} \;, \qquad K \;=\; \psi^{q}\;\bar{K} \;. \tag{2.2.18}$$

From the Hamiltonian constraint in equation (2.2.7), it seems simplest to leave the trace K conformally invariant, i.e. $q = 0$, and $K = \bar{K}$.

The divergence of any symmetric, trace-free tensor is related to its conformal divergence according to:

$$D_b A^{ab} \;=\; \psi^{-10}\;\bar{D}_b\left(\psi^{10+\alpha}\bar{A}^{ab}\right) \;, \tag{2.2.19}$$

which implies that the most natural choice of transformation for the *trace-free* part of the curvature is given by:

$$A^{ab} \;=\; \psi^{-10}\bar{A}^{ab} \;, \tag{2.2.20}$$

relating the divergence of A^{ab} to its equivalent in conformal terms, by the same factor:

$$D_b A^{ab} \;=\; \psi^{-10}\;\bar{D}_b\left(\bar{A}^{ab}\right) \;. \tag{2.2.21}$$

The conformally transformed extrinsic curvature is still given by 6 free components, since the conformal factor is associated with a component variable of the spatial metric.

Rewriting both sets of constraint equations, (2.2.7) and (2.1.43), in terms of the conformal metric, its associated terms, and the conformal trace-free part of the extrinsic curvature gives:

$$8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ab}\bar{A}^{ab} \;=\; -\,16\pi\psi^5\rho \;, \tag{2.2.22}$$

$$\bar{D}_b\;\bar{A}^{ab} - \frac{2}{3}\psi^6\bar{\gamma}^{ab}\bar{D}_b K \;=\; -\,8\pi\psi^{10}j^a \;. \tag{2.2.23}$$

Separating the *conformal* trace-free part of the extrinsic curvature into its transverse and longitudinal parts:

$$\bar{A}^{ab} \;=\; \bar{A}^{ab}_{TT} + \left(\bar{L}W\right)^{ab} \qquad \text{s.t.} \qquad \bar{D}_b\, \bar{A}^{ab}_{TT} \;=\; 0\ . \tag{2.2.24}$$

The divergence of $\bar{A}^{ab}$ is given by equation (2.2.16), in the equivalent conformal terms:

$$\begin{aligned}\bar{D}_b\bar{A}^{ab} \;&\equiv \bar{D}_b\left(\bar{L}W\right)^{ab} \\ &= \bar{D}^2W^a + \frac{1}{3}\bar{D}^a\left(\bar{D}_bW^b\right) + \bar{R}^a_bW^b \\ &:= \left(\bar{\triangle}_LW\right)^a\ ,\end{aligned} \tag{2.2.25}$$

from which the momentum constraint, in the form of (2.2.23), becomes:

$$\left(\bar{\triangle}_LW\right)^a - \frac{2}{3}\psi^6\bar{\gamma}^{ab}\bar{D}_bK \;=\; -\,8\pi\psi^{10}j^a\ . \tag{2.2.26}$$

Determination of Initial Conditions

To determine the initial conditions, according to the transverse trace-free decomposition, the 8 degrees of freedom are first specified by making choices for:

$$\bar{\gamma}_{ab}:\ \text{5 degrees of freedom,} \tag{2.2.27a}$$

$$K:\ \text{1 degree of freedom,} \tag{2.2.27b}$$

$$\bar{A}^{ab}_{TT}:\ \text{2 degrees of freedom.} \tag{2.2.27c}$$

with the momentum constraint, in the form (2.2.26), then solved for the vector W^a:

$$\left(\bar{\triangle}_LW\right)^a - \frac{2}{3}\,\psi^6\,\bar{\gamma}^{ab}\,\bar{D}_bK \;=\; -\,8\pi\,\psi^{10}j^a\ . \tag{2.2.28}$$

From the vector W^a and the freely specified transverse trace-free curvature $\bar{A}^{ab}_{TT}$, the conformal trace-free extrinsic curvature is found from (2.2.24):

$$\bar{A}^{ab} \;=\; \bar{A}^{ab}_{TT} + \left(\bar{L}W\right)^{ab}\ . \tag{2.2.29}$$

The Hamiltonian constraint (2.2.22) is then solved for the conformal factor ψ, using $\bar{A}^{ab}$ from above, and the freely specified $\bar{\gamma}_{ab}$ and K:

$$8\,\bar{D}^2\psi - \psi\,\bar{R} - \frac{2}{3}\,\psi^5K^2 + \psi^{-7}\bar{A}_{ab}\,\bar{A}^{ab} \;=\; -16\,\pi\,\psi^5\,\rho\ . \tag{2.2.30}$$

Finally, the fully physical spatial metric and extrinsic curvature are found from equations (2.2.1), (2.2.10) and (2.2.20):

$$\begin{aligned} \gamma_{ab} &= \psi^4 \, \bar{\gamma}_{ab} \ , \\ K_{ab} &= A_{ab} + \frac{1}{3} \, \gamma_{ab} \, K \ = \ \psi^{-2} \bar{A}_{ab} + \frac{1}{3} \, \gamma_{ab} \, K \ , \end{aligned} \tag{2.2.31}$$

which give the full set of initial conditions, satisfying Einstein's equations.

Determination of Initial Conditions *Without* a "TT" Tensor

Though trace-free tensors are easily constructed, by factoring out the computed trace (2.2.10), transverse trace-free tensors are not so easy to find. There is, however, a method of constructing symmetric TT-tensors from a trace-free tensor. Taking a symmetric trace-free tensor M^{ab}, a vector Y^a is defined such that:

$$\begin{aligned} \triangle_L Y^a \ &= \ D_b M^{ab} \ , \\ \Rightarrow \qquad D_b \left(M^{ab} - (LY)^{ab} \right) \ &= \ 0 \ , \end{aligned} \tag{2.2.32}$$

which gives a transverse trace-free tensor by construction:

$$A^{ab}_{TT} \ := \ M^{ab} - (LY)^{ab} \ . \tag{2.2.33}$$

The trace-free tensor A^{ab} can now be given by:

$$A^{ab} \ = \ M^{ab} - (LY)^{ab} + (LW)^{ab} \ , \tag{2.2.34}$$

joining equations (2.2.11) and (2.2.33). Since the conformal Killing form is linear with respect to its vector, a new vector V^a can be defined such that $V^a := W^a - Y^a$ so that A^{ab} can be given by:

$$A^{ab} \ = \ M^{ab} + (LV)^{ab} \ , \tag{2.2.35}$$

without having to define a transverse trace-free tensor.

Using this technique, a symmetric trace-free tensor $\bar{M}^{ab}$ can now be chosen as freely specified data instead of the transverse tensor $\bar{A}^{ab}_{TT}$ from (2.2.27c), giving the free data:

$$\bar{\gamma}_{ab} : \ \text{5 degrees of freedom,} \tag{2.2.36a}$$

$$K : \ \text{1 degree of freedom,} \tag{2.2.36b}$$

$$\bar{M}^{ab} : \ \text{2 degrees of freedom,} \tag{2.2.36c}$$

with the trace-free curvature part, in $\bar{M}^{ab}$, still containing 2 degrees of freedom.

The momentum constraint (2.2.28), with $\bar{M}^{ab}$ instead of $\bar{A}^{ab}_{TT}$, is given by:

$$\left(\bar{\triangle}_L V\right)^a + \bar{D}_b \bar{M}^{ab} - \frac{2}{3}\,\psi^6\,\bar{\gamma}^{ab}\,\bar{D}_b K \;=\; -\,8\pi\,\psi^{10} j^a\ , \tag{2.2.37}$$

and is solved for the vector V^a. The conformal trace-free curvature is then found by substituting conformal terms into (2.2.35):

$$\bar{A}^{ab} \;=\; \bar{M}^{ab} + \left(\bar{L}V\right)^{ab}\ , \tag{2.2.38}$$

and the spatial metric and extrinsic curvature found by solving the conformal Hamiltonian constraint (2.2.30) for ψ, and then (2.2.31) for both γ_{ab} and K_{ab} as before.

2.2.4 Conformal Thin Sandwich Decomposition

The *conformal 'thin sandwich'* approach, developed by *York* [55], allows the time evolution of the initial time slice Σ_0, to be used as input for the initial conditions. It therefore involves the lapse and shift (α and β^a), joining the gauge choices with the initial value problem. This means there are now 16 instead of 12 independent variables, with the constraints still fixing 4, but leaving 12 instead of 8 freely specifiable components.

The trace-free part of the time derivative of the spatial metric, is given by:

$$u_{ab} \;:=\; \gamma^{1/3}\partial_t\left(\gamma^{-1/3}\gamma_{ab}\right)\ , \tag{2.2.39}$$

and from the evolution equation for the spatial metric (2.1.45):

$$u^{ab} \;=\; 2\alpha A^{ab} + (L\beta)^{ab}\ . \tag{2.2.40}$$

The *conformal* time derivative is defined as the derivative of the conformal metric:

$$\bar{u}_{ab} \;:=\; \partial_t \bar{\gamma}_{ab}\ , \qquad \text{with} \qquad \bar{\gamma}^{ab}\bar{u}_{ab} \;:=\; 0\ . \tag{2.2.41}$$

The second equation above ensures that $\partial_t \ln \bar{\gamma} = 0$, and hence:

$$\bar{u}_{ab} \;=\; \psi^4 \bar{u}_{ab}\ , \qquad \text{and} \qquad (L\beta)^{ab} \;=\; \psi^{-4}\left(\bar{L}\beta\right)^{ab}\ . \tag{2.2.42}$$

The trace-free part of the extrinsic curvature is again conformally transformed according to the natural factor used in equation (2.2.20):

$$A^{ab} \;=\; \psi^{-10}\bar{A}^{ab}\ . \tag{2.2.43}$$

From this, and equation (2.2.42) above, the evolution equation for the spatial metric can be given by:

$$\bar{A}^{ab} = \frac{\psi^6}{2\alpha}\left(\bar{u}^{ab} - \left(\bar{L}\beta\right)^{ab}\right) . \tag{2.2.44}$$

To remove the conformal factor from (2.2.44), the lapse is conformally re-scaled by:

$$\bar{\alpha} := \psi^6\alpha , \tag{2.2.45}$$

and is known as the *densitised* lapse, since with the *natural* conformal factor $\psi = \gamma^{12}$, the conformal lapse is given by $\bar{\alpha} = \gamma^{1/2}\alpha$, relating the lapse to the spatial volume element. The evolution equation (2.2.44) can now be rewritten as:

$$\bar{A}^{ab} = \frac{1}{2\bar{\alpha}}\left(\bar{u}^{ab} - \left(\bar{L}\beta\right)^{ab}\right) , \tag{2.2.46}$$

removing the conformal factor from the equation altogether. This is then substituted into the conformal momentum constraint, from equation (2.2.23):

$$\left(\bar{\triangle}\beta\right)^a - \left(\bar{L}\beta\right)^{ab}\bar{D}_b \ln\bar{\alpha} = \bar{\alpha}\ \bar{D}_b\left(\frac{1}{\bar{\alpha}}\bar{u}^{ab}\right) - \frac{4}{3}\ \bar{\alpha}\ \psi^6\bar{D}^a K + 16\pi\bar{\alpha}\ \psi^{10} j^a . \tag{2.2.47}$$

Determination of Initial Conditions

For the conformal thin sandwich, the freely specifiable data is given by:

$$\bar{\gamma}_{ab} : \text{ 5 degrees of freedom,} \tag{2.2.48a}$$

$$\bar{u}_{ab} : \text{ 5 degrees of freedom,} \tag{2.2.48b}$$

$$K : \text{ 1 degree of freedom,} \tag{2.2.48c}$$

$$\bar{\alpha} : \text{ 1 degree of freedom,} \tag{2.2.48d}$$

with 12 degrees of freedom instead of the 8 for the conformal transverse trace-free decomposition. Using all of the free data given above ($\bar{\gamma}_{ab}$ for evaluating $\bar{D}$), the momentum constraint in the form (2.2.47), is solved for the lapse β^a:

$$\left(\bar{\triangle}\beta\right)^a - \left(\bar{L}\beta\right)^{ab}\bar{D}_b \ln\bar{\alpha} = \bar{\alpha}\ \bar{D}_b\left(\frac{1}{\bar{\alpha}}\bar{u}^{ab}\right) - \frac{4}{3}\ \bar{\alpha}\ \psi^6\bar{D}^a K + 16\ \pi\ \bar{\alpha}\ \psi^{10} j^a . \tag{2.2.49}$$

The conformal trace-free part of the extrinsic curvature can then be evaluated from equation (2.2.46):

$$\bar{A}^{ab} = \frac{1}{2\bar{\alpha}}\left(\bar{u}^{ab} - \left(\bar{L}\beta\right)^{ab}\right) , \tag{2.2.50}$$

which allows the Hamiltonian constraint (2.2.22) to be solved for the conformal factor

ψ, using also the freely specified $\bar{\gamma}_{ab}$ and K:

$$8\ \bar{D}^2\psi - \psi\ \bar{R} - \frac{2}{3}\ \psi^5 K^2 + \psi^{-7}\bar{A}_{ab}\ \bar{A}^{ab}\ =\ -16\pi\ \psi^5\rho\ . \tag{2.2.51}$$

The initial conditions, consisting of the physical spatial metric and extrinsic curvature, are again found from equations (2.2.1), (2.2.10) and (2.2.20):

$$\begin{aligned} \gamma_{ab}\ &= \psi^4\ \bar{\gamma}_{ab}\ , \\ K_{ab}\ &= \psi^{-2}\bar{A}_{ab} + \frac{1}{3}\ \gamma_{ab}\ K\ . \end{aligned} \tag{2.2.52}$$

The choices for the gauge variables are also made by the conformal thin sandwich approach. The physical lapse α is found from equation (2.2.45):

$$\alpha\ =\ \psi^{-6}\bar{\alpha}\ , \tag{2.2.53}$$

and the shift vector β^a from the solution of the momentum constraint (2.2.49) above.

2.2.5 Bowen-York Extrinsic Curvature

The Bowen-York extrinsic curvature was given by *Bowen & York* [10] in 1980, by solving the conformal transverse trace-free decomposition of section 2.2.3, for a vacuum space-like hypersurface, which is conformally flat and maximally sliced. As free initial conditions (2.2.36), with a symmetric tensor choice $\bar{M}^{ab}$ instead of $\bar{A}^{ab}_{TT}$, Bowen and York thus chose:

$$\bar{\gamma}_{ab}\ = \delta_{ab}\ , \tag{2.2.54a}$$

$$K\ = 0\ , \tag{2.2.54b}$$

$$\bar{M}^{ab}\ = 0\ . \tag{2.2.54c}$$

The adjusted momentum constraint of (2.2.37) then reduces to:

$$\begin{aligned} &\left(\bar{\triangle}_L V\right)^a + \bar{D}_b\cancelto{0}{\bar{M}^{ab}} - \frac{2}{3}\ \psi^6\ \bar{\gamma}^{ab}\ \bar{D}_b\cancelto{0}{K} = -\ 8\pi\ \psi^{10}\cancelto{0}{\bar{j}^a}, \\ \Leftrightarrow\qquad &\left(\bar{\triangle}_L V\right)^a\ =\ 0\ , \\ \Leftrightarrow\qquad &\bar{D}_b\left(\bar{L}V\right)^{ab}\ =\ 0\ , \end{aligned} \tag{2.2.55}$$

with solutions for the vector V^a given by:

$$V^a\ =\ -\frac{1}{4r}\left(7P^a + q^a q_b P^b\right) + \frac{1}{r^2}\ \epsilon^{abc}q_b J_c\ , \tag{2.2.56}$$

with q^a the unit normal of a sphere of constant radius. It can be shown that P^a and J^a represent linear and angular momenta respectively.

In the equation above, ϵ_{abc} is the Levi-Civita antisymmetric tensor, and is defined in a space-like hypersurface to be:

$$\epsilon_{abc} \;:=\; \begin{cases} \gamma^{1/2} & : \text{ for even permutations of } a,\ b,\ c, \\ -\gamma^{1/2} & : \text{ for odd permutations of } a,\ b,\ c, \\ 0 & : \text{ for any repeated indices} \end{cases} \tag{2.2.57}$$

with $\gamma^{1/2}$ representing the volume element. The upper index ϵ^{abc} can easily be seen, by raising indices with the spatial metric, to be equivalent to either $\gamma^{-1/2}$, $-\gamma^{-1/2}$ or 0.

The conformal trace-free curvature is found from (2.2.38), with V^a given by (2.2.56):

$$\bar{A}_{ab} \;=\; \cancelto{0}{\bar{M}_{ab}} + \left(\bar{L}V\right)_{ab} \;, \tag{2.2.58}$$

and since the trace of the extrinsic curvature was set to be zero (2.2.54b), the conformal extrinsic curvature is equal to the conformal trace-free curvature:

$$\bar{K}_{ab} \;=\; \bar{A}_{ab} \;=\; \left(\bar{L}V\right)_{ab} \;. \tag{2.2.59}$$

Two solutions to (2.2.59) are given by *Bowen & York* [10] (equations (9) and (10)):

$$\begin{aligned} \bar{K}^{\pm}_{ab} \;=&\; \frac{3}{2r^2}\; \left[P_a q_b + P_b q_a - (\bar{\gamma}_{ab} - q_a q_b) P^c q_c\right] \\ &\mp \frac{3a^2}{2\;r^4} \left[P_a q_b + P_b q_a + (\bar{\gamma}_{ab} - 5 q_a q_b) P^c q_c\right] \;, \end{aligned} \tag{2.2.60}$$

$$\bar{K}_{ab} \;=\; \frac{3}{r^3}\; \left(\epsilon_{acd}\; q_b + \epsilon_{bcd}\; q_a\right) q^c J^d \;, \tag{2.2.61}$$

with P^a and J^a representing the linear and angular momentum respectively, and a an arbitrary constant. These solutions are together known as the *Bowen-York* extrinsic curvature.

The physical metric and extrinsic curvature still require the conformal factor ψ to be found, by solving the conformal Hamiltonian constraint (2.2.30). This is reduced by the free data (2.2.54) to:

$$\begin{aligned} & 8\; \bar{D}^2\psi - \psi\; \cancelto{0}{\bar{R}} - \frac{2}{3}\; \psi^5 \cancelto{0}{K^2} + \psi^{-7} \bar{A}_{ab}\; \bar{A}^{ab} \;=\; -\,16\;\pi\;\psi^5\; \cancelto{0}{\rho} \;, \\ \Leftrightarrow \qquad & 8\; \bar{D}^2\psi + \psi^{-7} \bar{A}_{ab}\; \bar{A}^{ab} \;= 0 \;, \end{aligned} \tag{2.2.62}$$

though unfortunately this equation cannot be solved exactly, and it is only through numerical computation that solutions for the conformal factor ψ can be found.

2.3 Gauge Conditions

The gauge choices in the evolution of the $3+1$ formalism consist of the choices of the lapse α and shift β^a. As mentioned earlier, the lapse represents the advancement of *coordinate* time at different points of each time slice. The shift vector β^a then gives the change in coordinates between time slices, with respect to the normal observer.

Since Einstein's equations are coordinate invariant, there is no natural way of choosing the lapse and shift. However, different gauge choices have a big effect on the evolution of numerical codes. As a result, gauge conditions are sought that improve the effectiveness of the evolution equations.

Insight into the geometry of the space-time, and experience with numerical evolution have given the best direction in gauge choices. As such, a good choice of lapse and shift should, at the very least, avoid both physical and coordinate singularities and should make the mathematical equations to be solved as simple as possible.

The gauge conditions are generally broken into two groups. *Slicing conditions* involve the choice of lapse α, and hence the particular "slicing" of the space-time into time slices Σ_t. *Shift conditions* then involve the choice of the shift vector β^a, and hence the "shift" of spatial coordinates between each time slice. It has also been found, however, that the best choice of gauge conditions depends on a coupling of both the lapse and the shift.

This section is mainly based on *Smarr & York* [46, 47], with some advancements by *Balakrishna et al.* [6] and *Bona et al.* [9].

2.3.1 Geodesic Slicing

The most obvious choice, with regard to the simplicity of the evolution equations, is to let the coordinate observers coincide with the normal observers. This means choosing the lapse to be one, and the shift to vanish:

$$\alpha = 1 , \qquad \beta^a = 0 , \tag{2.3.1}$$

and is known as *geodesic* slicing, since the coordinate observers follow geodesics in the space-time manifold. Since the lapse is unitary, the time coordinates also coincide with proper time.

However, although geodesic slicing provides hugely simplified evolution equations, it does not satisfy the primary concern of avoiding singularities. Due to the attractive nature of gravity, all *free falling* observers will "fall" towards any nearby space-time singularity, and different coordinates will eventually merge.

Contracting the evolution equation for the extrinsic curvature (2.1.46), gives the evolution of its trace K:

$$\begin{aligned} \partial_t K &= \partial_t \, \gamma^{ab} K_{ab} \\ &= D^2\alpha - \alpha \left(K_{ab}K^{ab} + 4\pi(\rho + S) \right) + \beta^a D_a K \ . \end{aligned} \tag{2.3.2}$$

This reduces, under the geodesic slicing conditions (2.3.1), to:

$$\partial_t K \ = \ -K_{ab}K^{ab} - 4\pi \left(\rho + S\right) \ \leq \ 0 \ , \tag{2.3.3}$$

showing K to "shrink" monotonically in time. Hence, by equation (2.1.22):

$$K \ = \ \mathcal{L}_n \ln \gamma^{1/2} \ = \ \partial_t \ln \gamma^{1/2} \ , \tag{2.3.4}$$

the coordinate volume element accelerates to zero, leading to coordinate singularities.

2.3.2 Maximal and Constant Mean Curvature Slicing

It is clear from the problems with geodesic slicing, that the coordinate volume element is effected by the mean curvature K. Hence, to avoid coordinates approaching a singularity, the mean curvature can be chosen to keep the volume element from vanishing.

Once a choice for the mean curvature K and its time derivative $\partial_t K$, has been made, the evolution equation for K, (2.3.2), turns into an elliptic equation for the lapse:

$$\partial_t K \ = \ D^2\alpha - \alpha \left(K_{ab}K^{ab} + 4\pi(\rho + S) \right) + \beta^a D_a K \tag{2.3.5}$$

though the shift vector β^a must still be chosen.

Constant mean curvature (CMC) slicing fixes the value of the mean curvature K, by setting its time derivative to zero:

$$K \ = \ \text{constant} \ , \qquad \partial_t K \ = \ 0 \ . \tag{2.3.6}$$

The rate of change of the volume element can then be given by the choice of constant, to ensure that it doesn't shrink too quickly for the required evolution. CMC slicing reduces the elliptic equation for the lapse (2.3.5) to:

$$D^2\alpha \ = \ \alpha \left(K_{ab}K^{ab} + 4\pi(\rho + S) \right) - \beta^a D_a K \ . \tag{2.3.7}$$

The *maximal* slicing condition is given by choosing *both* the mean curvature and its derivative to be zero:

$$K \;=\; 0 \;, \qquad \partial_t K \;=\; 0 \;, \tag{2.3.8}$$

which *fixes* the coordinate volume elements, so that different coordinates can never coincide. This reduces equation (2.3.5) even further:

$$\begin{aligned} D^2\alpha \;&= \alpha\left(K_{ab}K^{ab} + 4\pi(\rho + S)\right) \\ &\equiv \alpha\left(R - 4\pi(3\rho - S)\right) \;. \end{aligned} \tag{2.3.9}$$

decoupling the shift vector β^a, leaving the elliptic equation for the lapse completely determined. Note, the Hamiltonian constraint has been combined with the first equation above, to give the second part, which is now entirely free from the extrinsic curvature.

The elliptic equation (2.3.9) is a second order partial differential equation for α, and so two boundary conditions must also be given to solve the equation. Outside of cosmological solutions, one such condition is usually given by assuming an asymptotically flat space-time manifold. This means assuming that the space-time manifold returns to that of flat Minkowski, far enough away from the area of interest. Since Minkowski space-time is flat, geodesic slicing is the most natural gauge condition. Hence, in asymptotically flat space-times, it is natural to choose the lapse so that:

$$\lim_{r_s \to \infty} \alpha \;=\; 1 \;, \tag{2.3.10}$$

where r_s is the Schwarzschild radial coordinate. The second boundary condition should also be informed by the geometry of the space-time manifold under study, e.g. the geometry close to black hole horizons.

Numerical error, however, can cause the mean curvature K to "drift" away from zero. The condition that the time derivative remain zero means that the curvature has no inclination to return to zero. As a result, *Balakrishna et al.* [6] have introduced a K-*driver* condition which drives the solution towards the maximal condition, stronger the further K gets from zero. Hence the K-driver condition is given by changing the time derivative of K to:

$$\partial_t K \;=\; -cK \;, \tag{2.3.11}$$

where the constant c can be used to control the strength of the condition. The equation for the lapse then becomes:

$$D^2\alpha \;=\; \alpha\left(K_{ab}K^{ab} + 4\pi(\rho + S)\right) - \beta^a D_a K + cK \;, \tag{2.3.12}$$

though with the shift condition returning to the equation.

2.3.3 Harmonic Coordinates and "1+Log" Slicing

Harmonic coordinates have many benefits, giving Einstein's equations as a set of wave equations, and giving the lapse and shift as solutions to a set of *hyperbolic* equations, which are easier to solve then elliptic equations, and take less computer time for simulations.

To begin, a contraction is taken of the 4-connection coefficients:

$$ {}^{(4)}\Gamma^{\alpha} \;=\; g^{\mu\nu}\;{}^{(4)}\Gamma^{\alpha}_{\mu\nu}\;, \tag{2.3.13} $$

which can be set equal to a *gauge source function.* For *harmonic coordinates* this function is set to zero:

$$ {}^{(4)}\Gamma^{\alpha} \;=\; g^{\mu\nu}\;{}^{(4)}\Gamma^{\alpha}_{\mu\nu} \;=\; 0\;. \tag{2.3.14} $$

Separating the time coordinate t from the spatial coordinates a, b, and substituting in for the 4-metric in the $3+1$ format (2.1.14), the lapse and shift are given by:

$$ \begin{aligned} \left(\partial_t - \beta^b\partial_b\right)\alpha \;&=\; -\,\alpha^2 K\;, \\ \left(\partial_t - \beta^b\partial_b\right)\beta^a \;&=\; -\,\alpha^2\left(\gamma^{ab}\partial_b \ln\alpha + \gamma^{bc}\Gamma^{a}_{bc}\right)\;, \end{aligned} \tag{2.3.15} $$

i.e. a set of hyperbolic equations, known as the *de Donder* gauge in *Smarr & York* [47].

Harmonic *slicing* is given by setting only the time part of (2.3.13) to zero ${}^{(4)}\Gamma^{t} = 0$. When the shift vector is also set to zero, the lapse is found from the equation:

$$ \begin{aligned} \partial_t\alpha \;&=\; \alpha^2 K \\ \Rightarrow \qquad \alpha \;&=\; C(x^i)\;\gamma^{1/2}\;, \qquad \beta^a = 0\;, \end{aligned} \tag{2.3.16} $$

with $C(x^i)$ a constant of integration, dependent on the spatial coordinates alone. This slicing condition is equivalent to keeping the densitised lapse (2.2.45) constant, showing a similarity to geodesic slicing.

This leads to a generalised form of the harmonic slicing condition, known as the *Bona-Massó* family of slicing conditions (see *Bona, Massó, et al.* [9]), given by introducing a positive function $f(\alpha)$:

$$ \partial_t\alpha \;=\; -\alpha^2 f(\alpha)\;K\;, \tag{2.3.17} $$

which reduces to harmonic slicing for $f = 1$, and can be considered to reduce to geodesic slicing with $f = 0$. A particularly useful condition is given by taking $f = 2/\alpha$. This condition is known as the "*1+log*" slicing, and gives an algebraic equation for the lapse:

$$ \alpha \;=\; 1 + \ln\gamma\;, \qquad \beta^a \;=\; 0\;, \tag{2.3.18} $$

making it extremely easy to implement. The 1+log slicing also has better singularity avoidance than harmonic slicing.

Allowing for a non-zero shift vector, the 1+log slicing can be given by:

$$\left(\partial_t - \beta^b \partial_b\right) \alpha \;=\; -2\alpha K \;, \tag{2.3.19}$$

which, though not algebraic, still provides an effective hyperbolic slicing condition.

2.3.4 Minimal Distortion Shift Conditions

When making choices for the propagation of the spatial coordinates, via the shift vector β^a, it is important not to have coordinates "stretch" too far. This causes large areas of interest to be uncovered by data points, and can cause difficulties in certain simulations. To combat this, the *minimal distortion* condition has been developed, see *Smarr & York* [46].

The trace-free part of the time derivative of the spatial metric u_{ab}, from equation (2.2.39), is first decomposed into its transverse and longitudinal parts:

$$u_{ab} \;=\; u_{ab}^{TT} + u_{ab}^{L} \;. \tag{2.3.20}$$

The longitudinal part can then be given by:

$$u_{ab}^{L} \;=\; D_a X_b + D_b X_a - \frac{2}{3}\gamma_{ab} D^c X_c \;=\; (LX)^{ab} \;, \tag{2.3.21}$$

for some vector X^a, similar to equation (2.2.9) for the longitudinal part of K_{ab}.

Taking the natural conformal factor (2.2.3), $\psi = \gamma^{1/12}$, u_{ab}^{L} can be seen as the Lie derivative of the tensor density $\bar{\gamma}_{ab}$:

$$u_{ab}^{L} \;=\; \psi^4 \bar{u}_{ab}^{L} \;=\; \gamma^{1/3} \mathcal{L}_X \bar{\gamma}_{ab} \;, \tag{2.3.22}$$

similar to (2.2.14), for the longitudinal part of the trace-free curvature A_{ab}^{L}. Hence u_{ab}^{L} can be considered to arise from a coordinate change by X^a. Choosing $u_{ab}^{L} = 0$, eliminates the time-dependent coordinate changes, and gives a divergence-free u_{ab}:

$$D^b u_{ab} \;=\; 0 \;. \tag{2.3.23}$$

Equation (2.2.40), from the spatial metric evolution equation:

$$u_{ab} \;=\; 2\alpha A_{ab} + (L\beta)_{ab} \ , \tag{2.3.24}$$

is then divergence-free on both sides, when equation (2.3.23) is satisfied, giving:

$$D^b \,(L\beta)_{ab} \;=\; -\, 2D^b \,(\alpha A_{ab}) \ . \tag{2.3.25}$$

Combining the momentum constraint (2.1.43) with equation (2.3.25) above gives the *minimal distortion* condition:

$$(\triangle_L \beta)^a \;=\; -\, 2A^{ab} D_b \alpha - \frac{4}{3}\alpha\gamma^{ab} D_b K + 16\pi\alpha j^a \ . \tag{2.3.26}$$

In conformal terms, the minimal distortion condition is given by:

$$\left(\bar{\triangle}_L \beta\right)^a - \left(\bar{L}\beta\right)^{ab} \bar{D}_b \ln\psi^6 \;=\; -\, 2\ \psi^{-6} \bar{A}^{ab} \bar{D}_b \alpha - \frac{4}{3}\alpha\bar{\gamma}^{ab} \bar{D}_b K + 16\pi\psi^4 \alpha j^a \ . \tag{2.3.27}$$

Unfortunately, both equations (2.3.26) and (2.3.27) consist of a set of elliptic equations which, as noted before, are difficult to solve and take up a lot of computer simulation time. As a result, like the maximal slicing condition, attempts have been made to come up with approximate conditions that are easier to solve.

Defining the *conformal connection functions* by a contraction of the conformal connection coefficients:

$$\bar{\Gamma}^a \;:=\; \bar{\gamma}^{bc} \bar{\Gamma}^a_{bc} \ , \tag{2.3.28}$$

and again, taking the natural conformal factor $\psi = \gamma^{1/12}$, the conformal connection functions reduce, in cartesian coordinates, to:

$$\bar{\Gamma}^a \;\equiv\; -\partial_b \bar{\gamma}^{ab} \ . \tag{2.3.29}$$

An alternate condition to minimal distortion, is given by setting the time derivative of the connection functions to zero:

$$\partial_t \bar{\Gamma}^a \;=\; 0 \ , \tag{2.3.30}$$

known as Γ-*freezing.* This condition can be shown to be equivalent to:

$$\partial_b \ \bar{u}^{ab} \;=\; 0 \ , \tag{2.3.31}$$

which is very similar to the minimal distortion condition (2.3.23). However, Γ-freezing still results in a complicated set of elliptic equations.

Success has been found in approximating Γ-freezing, with a Γ-*driver* condition:

$$\partial_t \beta^a \;=\; k\left(\partial_t \bar{\Gamma}^a + \eta \bar{\Gamma}^a\right) \;, \tag{2.3.32}$$

with k and η positive constants. Taking the Γ-freezing condition initially, gives a set of parabolic equations. A hyperbolic Γ-driver can also be constructed with:

$$\partial_t \beta^a \;=\; \frac{3}{4} B^a \;, \qquad \partial_t B^a \;=\; \partial_t \bar{\Gamma}^a - \eta B^a \;, \tag{2.3.33}$$

with η typically of order $(2M)^{-1}$.

2.4 BSSN Reformulation

There have been a number of reformulations of the ADM 3 + 1 constraint and evolution equations, of which the BSSN (Baumgarte, Shapiro, Shibata, Nakamura) is one. These reformulations tend to be more stable and flexible than the original ADM equations, and benefit from advances in both initial and gauge conditions, see for example *Baumgarte & Shapiro* [7].

2.4.1 Conformal Transformation

The BSSN reformulation begins by conformally transforming the spatial metric, using the *natural* conformal factor, so that the conformal metric has a determinant of one:

$$\gamma_{ab} \;=\; \psi^4 \bar{\gamma}_{ab} \;, \qquad \text{s.t.} \qquad \psi \;=\; \gamma^{1/12} \;. \tag{2.4.1}$$

The conformal factor is then changed to an exponential form:

$$\begin{aligned} \psi \;=\; e^{\phi} \;, \qquad \text{s.t.} \qquad \gamma_{ab} &= e^{4\phi}\bar{\gamma}_{ab} \;, \\ \phi &= \frac{1}{12}\ln\gamma \;, \end{aligned} \tag{2.4.2}$$

allowing later equations to take easier forms. From here on, the conformal metric will be assumed to be in cartesian type coordinates.

The extrinsic curvature is broken into its trace and trace-free parts:

$$K_{ab} \;=\; A_{ab} + \frac{1}{3}\gamma_{ab}K \;, \tag{2.4.3}$$

and the trace-free part conformally transformed according to:

$$A_{ab} \;=\; e^{4\phi}\bar{A}_{ab} \;. \tag{2.4.4}$$

Contractions of the evolution equations (2.1.45) and (2.1.46) give equations for the trace of spatial metric and extrinsic curvature:

$$\partial_t \ln\gamma^{1/2} \;= \alpha K + D_a\beta^a \;, \tag{2.4.5}$$

$$\partial_t K \;= D^2\alpha - \alpha\left(K_{ab}K^{ab} + 4\pi(\rho + S)\right) + \beta^a D_a K \;. \tag{2.4.6}$$

Note that the evolution equation for K has already been given in equation (2.3.2). The evolution equation for the conformal factor ϕ can now be given:

$$\partial_t\phi \;=\; \frac{1}{6}\left(\alpha K + \partial_a\beta^a\right) + \beta^a\partial_a\phi \;. \tag{2.4.7}$$

The evolution equations for the metric and extrinsic curvature are found directly from the ADM equations. The evolution for the trace-free part of the curvature is found by subtracting equation (2.4.6) from (2.1.46). In conformal terms, these are then given by:

$$
\begin{aligned}
d_t\bar{\gamma}_{ab} &= 2\alpha\bar{A}_{ab}, && (2.4.8)\\
d_t\bar{A}_{ab} &= e^{-4\phi}\left\{D_aD_b\alpha - \alpha R_{ab} + 8\pi\alpha S_{ab}\right\}^{TF} - \alpha\left(K\bar{A}_{ab} - 2\bar{A}_{ac}\bar{A}^c_b\right) \ , && (2.4.9)
\end{aligned}
$$

where the derivative $d_t = \partial_t - \mathcal{L}_\beta$, and the superscript TF means the trace-free part is to be taken, for everything inside the brackets.

2.4.2 Conformal Connection Functions

A new set of variables are introduced, to eliminate mixed partial derivatives in the Ricci curvature. These are the *conformal connection functions* of equation (2.3.28):

$$
\bar{\Gamma}^a \ := \ \bar{\gamma}^{bc}\bar{\Gamma}^a_{bc} \ = \ -\partial_b\bar{\gamma}^{ab} \ , \tag{2.4.10}
$$

with the last part specifically due to $\bar{\gamma} = 1$, see equation (2.3.29).

These functions simplify the conformal Ricci tensor to:

$$
\begin{aligned}
\bar{R}_{ab} \ = \ &-\frac{1}{2}\left(\bar{\gamma}^{cd}\partial_c\partial_d\bar{\gamma}_{ab} + \bar{\gamma}_{ca}\partial_b\bar{\Gamma}^c + \bar{\gamma}_{cb}\partial_a\bar{\Gamma}^c + \bar{\Gamma}^c\bar{\Gamma}_{abc} + \bar{\Gamma}^c\bar{\Gamma}_{bac}\right)\\
&+\bar{\gamma}^{cd}\left(\bar{\Gamma}^e_{ca}\bar{\Gamma}_{bed} + \bar{\Gamma}^e_{cb}\bar{\Gamma}_{aed} + \bar{\Gamma}^e_{ad}\bar{\Gamma}_{ecb}\right) \ ,
\end{aligned} \tag{2.4.11}
$$

where the only second derivatives arising should give a convenient wave equation. This can now be used in the evolution equation for the trace-free part of the extrinsic curvature (2.4.9).

With the conformal connection coefficients now acting as variables in the evolution equation for $\bar{A}_{ab}$, the evolution equations for the functions themselves need to be found. This is done by directly using the evolution of the conformal metric (2.4.5):

$$
\begin{aligned}
\partial_t\bar{\Gamma}^a \ &= \ -\partial_t\partial_b\bar{\gamma}^{ab} \ = \ -\partial_b\partial_t\bar{\gamma}^{ab}\\
&= \ -\partial_b\left(2\alpha\bar{A}^{ab} - \mathcal{L}_\beta\bar{\gamma}^{ab}\right)\\
&= \ -2\left(\alpha\partial_b\bar{A}^{ab} + \bar{A}^{ab}\partial_b\alpha\right) + \partial_b\mathcal{L}_\beta\bar{\gamma}^{ab} \ .
\end{aligned} \tag{2.4.12}
$$

The divergence of $\bar{A}^{ab}$ can be replaced, by a reformulation of the momentum constraint:

$$
\begin{aligned}
d_t\bar{\Gamma}^a \ = \ &\bar{\gamma}^{bc}\partial_b\partial_c\beta^a + \frac{1}{3}\bar{\gamma}^{ab}\partial_b\partial_c\beta^c + 2\bar{A}^{ab}\partial_b\alpha\\
&+2\alpha\left(-\bar{\Gamma}^a_{bc}\bar{A}^{bc} - 6\bar{A}^{ab}\partial_b\phi + \frac{2}{3}\bar{\gamma}^{ab}\partial_bK - 8\pi\bar{\gamma}^{ab}j_b\right) \ .
\end{aligned} \tag{2.4.13}
$$

2.4.3 BSSN Equations

In their final form, the BSSN system of equations is given by decomposing the initial conditions of the spatial metric and extrinsic curvature by the equations:

$$\gamma_{ab} = e^{4\phi}\bar{\gamma}_{ab} \ , \tag{2.4.14}$$

$$K_{ab} = e^{4\phi}\bar{A}_{ab} + \frac{1}{3}\gamma_{ab}K \ , \tag{2.4.15}$$

$$\phi = \frac{1}{12}\ln\gamma \ . \tag{2.4.16}$$

There are now three sets of constraint equations, the Hamiltonian and momentum constraints from the $3+1$ formalism, and the conformal connection coefficients:

$$\bar{\gamma}^{ab}\bar{D}_a\bar{D}_b e^{\phi} + \frac{1}{8}e^{\phi}\bar{R} + \frac{1}{8}e^{5\phi}\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{12}e^{5\phi}K^2 = -2\pi e^{5\phi}\rho \ , \tag{2.4.17}$$

$$\bar{D}_b\left(e^{6\phi}\bar{A}^{ab}\right) - \frac{2}{3}e^{6\phi}\bar{D}^a K = -8\pi e^{6\phi}j^a \ , \tag{2.4.18}$$

$$\bar{\Gamma}^a := \bar{\gamma}^{bc}\bar{\Gamma}^a_{bc} = -\partial_b\bar{\gamma}^{ab} \ . \tag{2.4.19}$$

The evolution equations are then given by:

$$d_t\phi = \frac{1}{6}\alpha K \ , \tag{2.4.20}$$

$$d_t\bar{\gamma}_{ab} = 2\alpha\bar{A}_{ab}, \tag{2.4.21}$$

$$d_t K = D^2\alpha - \alpha\left(K_{ab}K^{ab} + 4\pi(\rho+S)\right) \ , \tag{2.4.22}$$

$$d_t\bar{A}_{ab} = e^{-4\phi}\left\{D_aD_b\alpha - \alpha R_{ab} + 8\pi\alpha S_{ab}\right\}^{TF} - \alpha\left(K\bar{A}_{ab} - 2\bar{A}_{ac}\bar{A}^c_b\right) \ , \tag{2.4.23}$$

$$\begin{aligned} d_t\bar{\Gamma}^a &= \bar{\gamma}^{bc}\partial_b\partial_c\beta^a + \frac{1}{3}\bar{\gamma}^{ab}\partial_b\partial_c\beta^c + 2\bar{A}^{ab}\partial_b\alpha \\ &\quad + 2\alpha\left(-\bar{\Gamma}^a_{bc}\bar{A}^{bc} - 6\bar{A}^{ab}\partial_b\phi + \frac{2}{3}\bar{\gamma}^{ab}\partial_b K - 8\pi\bar{\gamma}^{ab}j_b\right) \ . \end{aligned} \tag{2.4.24}$$

with $d_t := \partial_t - \mathcal{L}_\beta$, and noting that ϕ is a tensor density of weight $\frac{1}{6}$, and $\bar{\gamma}_{ab}$, $\bar{A}_{ab}$ and $\bar{\Gamma}^a$ are all tensor densities of weight $-\frac{2}{3}$.

Chapter 3

Axially-Symmetric Transverse Trace-Free Tensors

In the conformal transverse trace-free decomposition, described in section 2.2.3, transverse trace-free tensors play an important role, with the "TT" part of the conformal extrinsic curvature acting as freely specified data (2.2.27c), for the construction of initial conditions.

It can be seen from the development of the transverse and trace-free decompositions of the extrinsic curvature, in section 2.2.2, that in any particular coordinate system, $\bar{A}^{ab}_{TT}$ only contains 2 independent component choices, if it is to be symmetric, transverse and trace-free.

Once a spatial coordinate system has been chosen, the 6 components of the symmetric tensor $\bar{A}^{ab}_{TT}$ can be considered subject to transverse and trace-free "constraint" equations:

$$\bar{A}^{ab}_{TT} \; , \qquad \text{s.t.} \qquad \bar{D}_b \, \bar{A}^{ab}_{TT} = 0 \; , \tag{3.0.1a}$$

$$\bar{\gamma}_{ab} \, \bar{A}^{ab}_{TT} = 0 \; . \tag{3.0.1b}$$

Deciding which 2 components to choose freely, and which to constrain can be a similar problem to the original initial condition problem of section 2.2. Also, the divergence-free equation (3.0.1a) consists of a set of coupled partial differential equations for the components of $\bar{A}^{ab}_{TT}$, and can be quite difficult to solve without additional constraints and boundary conditions.

This chapter is concerned with finding an expression for transverse trace-free tensors (of the same type as $\bar{A}^{ab}_{TT}$), which depend on 2 scalar potentials, with the added condition of axial symmetry. This extra condition to the constraints in (3.0.1), allows the number of derivative components in each of the divergence equations to be reduced, thereby helping to decouple them.

Section 3.1 first gives an outline of previous work by others, on axially symmetric transverse trace-free tensors. Tensor expressions for a flat metric are derived in section 3.2, in both spherical and cylindrical coordinates, with a comparison made between the scalar potentials for each. This is then extended to a more general axially symmetric metric, in this case the Brill metric, in section 3.3.

The expression for the flat space tensors is compared with the Bowen-York extrinsic curvature in section 3.4, showing a particular choice of scalar potentials to give the Bowen-York curvature. Choices of scalar potentials, for tensors that are regular at the origin, are then investigated in section 3.5, and conditions for a spherically symmetric product $T_{ab}T^{ab}$ outlined in section 3.6.

3.1 Previous Work

Two main papers in the literature deal with similar tensors to those studied in this chapter. The first, by *Baker & Puzio* [5] from 1999, reduces part of the extrinsic curvature to a choice of scalar potential, for a general axially-symmetric spatial metric. The second, *Sergio Dain* [19] from 2001, generalizes the results of *Baker & Puzio* [5] to a coordinate independent representation, also using a time-symmetry result from *Brandt & Seidel* [12] of 1996.

3.1.1 *Baker & Puzio*

Baker and Puzio [5] assume a spacelike hypersurface with a Killing vector field, corresponding to axial symmetry, given by $\partial/\partial\phi$, with ϕ a spatial coordinate. The coordinate plane for the remaining two spatial coordinates can be given as conformally equivalent to a flat 2-surface, giving a spatial metric, in the notation of this thesis, as:

$$\gamma_{ij}\; dx^i dx^j \;=\; A(x,y)(dx^2+dy^2)+B(x,y)d\phi^2\;. \tag{3.1.1}$$

This metric can be considered to be conformally related to the Brill metric, which is used at a later stage, see section 3.3.1.

The source-free Einstein constraints are then given as:

$$D^j K_{ij} \;=\; \partial_i(\gamma^{jk}K_{jk})\;, \tag{3.1.2}$$

$$(\gamma^{ac}\gamma^{bd}-\gamma^{ab}\gamma^{cd})\;K_{ab}K_{cd} \;=\; {}^{(3)}R\;. \tag{3.1.3}$$

The ϕ part of the momentum constraint (3.1.3), evaluated with respect to the above metric (3.1.1), is reduced in *Baker & Puzio* [5] to the following equation:

$$\partial^x(B^{1/2}K_{x\phi})+\partial^y(B^{1/2}K_{y\phi}) \;=\; 0, \tag{3.1.4}$$

from which, they write $K_{x\phi}$ and $K_{y\phi}$ in terms of a "potential function" $u(x,y)$:

$$K_{x\phi} \;=\; B^{-1/2}\partial_y u\;, \qquad K_{y\phi} \;=\; -B^{-1/2}\partial_x u\;. \tag{3.1.5}$$

Raising the indices in (3.1.5), for later comparison gives:

$$K^{x\phi} \;=\; \frac{\partial_y u}{AB^{3/2}}\;, \qquad K^{y\phi} \;=\; -\frac{\partial_x u}{AB^{3/2}}\;. \tag{3.1.6}$$

3.1.2 *Brandt & Seidel*

Brandt and Seidel [12] assume their initial slice to be maximal, see (2.3.8), and their space-time manifold to have a "time-rotation" symmetry about this initial slice. This equates to an invariance of the space-time metric under a transformation:

$$(t,\ \phi) \quad \rightarrow \quad (-t,\ -\phi)\ . \tag{3.1.7}$$

This time-rotation symmetry, from equations (15) and (16) of *Brandt & Seidel* [12], gives the extrinsic curvature to be of the form:

$$K_{ab} \ = \ \psi^{-2}\hat{H}_{ab} \ = \ \psi^{-2}\begin{pmatrix} 0 & 0 & \hat{H}_E\ \sin^2\theta \\ 0 & 0 & \hat{H}_F\ \sin\theta \\ \hat{H}_E\ \sin^2\theta & \hat{H}_F\ \sin\theta & 0 \end{pmatrix}\ , \tag{3.1.8}$$

in terms of the spherical-polar type coordinates (η, θ, ϕ), with ψ representing the conformal factor. Brandt and Seidel then show in equation (17) of [12], that the only non-trivial momentum constraint is given by the ϕ component:

$$\partial_\eta \hat{H}_E\ \sin^3\theta + \partial_\theta\left(\hat{H}_F\ \sin^2\theta\right) \ = \ 0\ . \tag{3.1.9}$$

Note that this is the same part of the momentum constraint that is given by a "potential function" in *Baker & Puzio* [5], see equation (3.1.4).

3.1.3 *Sergio Dain*

Sergio Dain [19] is a study of "head-on collisions for two Kerr-like black holes", using initially maximally sliced Kerr black holes, which satisfy the "time-rotation" symmetry (3.1.7) of *Brandt & Seidel* [12].

Equations (16) and (17) of *Brandt & Seidel* [12], see (3.1.8), can then be used to reduce the conformal extrinsic curvature of *Baker & Puzio* [5], to those terms given by the "potential function" of (3.1.5), i.e. in the (x, y, ϕ) coordinates of *Baker & Puzio* [5], taking (3.1.1) as the conformal metric:

$$K_{ab} \ = \ \psi^{-2}\begin{pmatrix} 0 & 0 & B^{-1/2}\partial_y u \\ 0 & 0 & -B^{-1/2}\partial_x u \\ B^{-1/2}\partial_y u & -B^{-1/2}\partial_x u & 0 \end{pmatrix}\ . \tag{3.1.10}$$

These results are generalized in *Dain* [19], to a coordinate independent form.

A spatial metric γ_{ab} is first assumed, with a hypersurface orthogonal Killing vector η^a, whose norm is given by $\eta = \gamma_{ab}\, \eta^a \eta^b$. The extrinsic curvature tensor is then defined by:

$$K^{ab} \;=\; \frac{1}{\eta}\left(S^a \eta^b + S^b \eta^a\right) \;, \tag{3.1.11}$$

where S^a satisfies:

$$\mathcal{L}_\eta S^a \;= 0 \;, \tag{3.1.12a}$$

$$S^a \eta_a \;= 0 \;, \tag{3.1.12b}$$

$$D_a S^a \;= 0 \;. \tag{3.1.12c}$$

The curvature tensor K^{ab} is said to be transverse and trace-free, due to equations (3.1.12), the Killing equation for η^a, and the fact that η^a is hypersurface orthogonal. The Killing equation is given by (1.2.74), and the equation for η^a being hypersurface orthogonal is given in *Dain* [19]:

$$D_a \eta_b + D_b \eta_a \;=\; 0 \;, \tag{3.1.13a}$$

$$D_a \eta_b \;=\; \frac{1}{2}\left(\eta_b D_a \ln \eta - \eta_a D_b \ln \eta\right) \;. \tag{3.1.13b}$$

The solution to (3.1.11) is given in [19], in terms of a "scalar potential" ω, by:

$$S^a \;=\; \frac{1}{\eta}\epsilon^{abc}\eta_b D_c \omega \;, \qquad \text{s.t.} \qquad \mathcal{L}_\eta \omega \;=\; 0 \;, \tag{3.1.14}$$

where, ϵ_{abc} is the Levi-Civita antisymmetric tensor given by equation (2.2.57).

Substituting (3.1.14) into (3.1.11), gives the extrinsic curvature tensor itself, in terms of the scalar potential ω:

$$K^{ab} \;=\; \epsilon^{ajk}\, \frac{\eta_j \eta^b}{\eta^2}\, D_k \omega \;+\; \epsilon^{bjk}\, \frac{\eta_j \eta^a}{\eta^2}\, D_k \omega \;, \tag{3.1.15}$$

giving a coordinate-independent representation of an axially-symmetric, transverse, trace-free tensor.

3.1.4 Transverse and Trace-Free Nature of *Sergio Dain*

The extrinsic curvature K^{ab} of *Dain* [19], (3.1.15), is tested to be transverse and trace-free, using equations (3.1.12) and (3.1.13).

Before taking the divergence of K^{ab}, a number of relations need to be derived. To begin, the vanishing of the Lie derivative of S^a with respect to the Killing vector field η^a, (3.1.12a), is explicitly given by equation (1.2.48):

$$\begin{aligned} 0 &= \mathcal{L}_\eta S^a \\ &= \eta^b D_b S^a - S^b D_b \eta^a \ , \end{aligned}$$

$$\Leftrightarrow \qquad \eta^b D_b S^a \ = \ S^b D_b \eta^a \ . \tag{3.1.16}$$

Raising one of the indices in the Killing equation for η^a, (3.1.13a):

$$D_a\eta_b + D_b\eta_a \ = 0 \ ,$$

$$\Leftrightarrow \qquad \gamma_{bc} D_a \eta^c + \gamma_{bc} D^c \eta_a \ = 0 \ ,$$

$$\Leftrightarrow \qquad D_a\eta^a \ = \ -D_a\eta^a \ = 0 \ . \tag{3.1.17}$$

The derivative of the norm of the Killing vector can be expanded by:

$$\begin{aligned} D_a\eta &= D_a\eta^c\eta_c \\ &= \eta^c D_a\eta_c + \eta_c D_a \eta^c \\ &= \eta^c D_a\eta_c + \eta_c\gamma^{cd} D_a\eta_d \\ &= 2\ \eta^b D_a\eta_b \ , \end{aligned} \tag{3.1.18}$$

and finally, equation (3.1.13b), for the hypersurface orthogonality of the Killing vector η^a, can then be expanded by:

$$D_a\eta_b \ = \frac{1}{2}\left(\eta_b D_a \ln\eta - \eta_a D_b \ln\eta\right) \ ,$$

$$\begin{aligned} \Leftrightarrow \qquad 2\ \eta^a\eta^b D_a\eta_b \ &= \eta^a\eta^b\eta_b\frac{D_a\eta}{\eta} - \eta^a\eta^b\eta_a\frac{D_b\eta}{\eta} \ , \\ \Leftrightarrow \qquad \cancel{\eta^a D_a}\eta \ &= \cancel{\eta^a D_a}\eta - \eta^b D_b\eta \ , \end{aligned}$$

$$\Leftrightarrow \qquad \eta^b\ D_b\ \eta \ = 0 \ , \tag{3.1.19}$$

with equation (3.1.18) used to substitute for $2\ \eta^b D_a\eta_b$, in the third equation above.

The divergence of K^{ab} can now be taken, from (3.1.11):

$$\begin{aligned}
D_a K^{ab} &= D_a \frac{1}{\eta}\left(S^a \eta^b + S^b \eta^a\right) \\
&= -\frac{\partial_a \eta}{\eta^2}\left(S^a \eta^b + S^b \eta^a\right) \\
&\quad + \frac{1}{\eta}\left(\eta^b \cancelto{0}{D_a S^a} + S^a D_a \eta^b + \eta^a D_a S^b + S^b \cancelto{0}{D_a \eta^a}\right) \\
&= -\frac{\partial_a \eta}{\eta^2}\left(S^a \eta^b + S^b \eta^a\right) + \frac{1}{\eta}\left(S^a D_a \eta^b + S^a D_a \eta^b\right) ,
\end{aligned} \tag{3.1.20}$$

using the relations (3.1.12c) and (3.1.17) to cancel the terms in the second line above, and the relation (3.1.16) substituted between the second and third lines. Multiplying both sides of (3.1.20) by η then gives:

$$\begin{aligned}
\eta\, D_a K^{ab} &= -\frac{\partial_a \eta}{\eta}\left(S^a \eta^b + S^b \eta^a\right) + 2 S^a D_a \eta^b \\
&= -S^a \eta^b D_a \ln \eta - \frac{1}{\eta} S^b \cancelto{0}{\eta^a D_a \eta} + 2 S^a D_a \eta^b \\
&= -\cancel{S^a \eta^b D_a \ln \eta} + \cancel{S^a \eta^b D_a \ln \eta} - \cancelto{0}{S^a \eta_a} D^b \ln \eta \\
&= 0 ,
\end{aligned}$$

$$\Leftrightarrow \qquad D_a K^{ab} = 0 , \tag{3.1.21}$$

with relation (3.1.19) used for the cancelation in the second line, (3.1.13b) used between the second and third lines, and (3.1.12b) used for the cancelation in the third line.

The curvature tensor K^{ab} is easily shown to be trace-free as well, since:

$$\begin{aligned}
K &= \gamma_{ab} K^{ab} \\
&= \frac{\gamma_{ab}}{\eta}\left(S^a \eta^b + S^b \eta^a\right) \\
&= \frac{1}{\eta}\left(\cancelto{0}{S^a \eta_a} + \cancelto{0}{S^b \eta_b}\right) \\
&= 0 ,
\end{aligned} \tag{3.1.22}$$

with the canceling due to equation (3.1.12b).

Thus, by equations (3.1.21) and (3.1.22), the tensor K^{ab}, defined by (3.1.11) and equations (3.1.12), in a spacelike hypersurface with a Killing vector field satisfying equations (3.1.13), is both transverse and trace-free.

3.1.5 Testing of Solution S^a

The solution given for S^a in equation (3.1.14), should satisfy all of the conditions (3.1.12), to be a valid solution for (3.1.11).

The Lie derivative of S^a (3.1.12a), is given by equation (1.2.48) for the Lie derivative of a vector:

$$\begin{aligned}
\mathcal{L}_\eta S^a &= \mathcal{L}_\eta \left(\frac{1}{\eta} \epsilon^{abc} \eta_b D_c \omega \right) \\
&= \epsilon^{abc} \left(\eta_b \; D_c\omega \; \mathcal{L}_\eta \frac{1}{\eta} + \frac{1}{\eta} \; D_c \omega \; \mathcal{L}_\eta \eta_b + \frac{1}{\eta} \; \eta_b \; \mathcal{L}_\eta \; D_c \omega \right) \\
&= \frac{1}{\eta} \; \epsilon^{abc} \Big(- \eta_b \; D_c \omega \; \frac{1}{\eta} \; \mathcal{L}_\eta \eta \\
&\qquad + D_c \omega \; \left(\eta^d D_d \eta_b + \eta_d D_b \eta^d \right) \\
&\qquad + \eta_b \; \left(\eta^d D_d D_c w + D_d w \cdot D_c \eta^d \right) \Big) \; ,
\end{aligned} \tag{3.1.23}$$

with the first term given by the chain rule for the Lie derivative, and the definition of the Lie derivative of a covector (1.2.51) used for the other two. The first term in (3.1.23) can be canceled, see equation (3.1.19), with the Killing equation (3.1.13a) applied to the middle term, and the equivalence of mixed partial derivatives used for the third:

$$\begin{aligned}
\mathcal{L}_\eta S^a &= \frac{1}{\eta} \; \epsilon^{abc} \Big(- \frac{1}{\eta} \; \eta_b \; D_c \omega \; \cancelto{0}{\eta^d D_d \eta} \\
&\qquad + D_c \omega \; \left(\eta^d D_d \eta_b - \eta_d D^d \eta_b \right) \\
&\qquad + \eta_b \; \left(\eta^d D_c D_d w + D_d w \cdot D_c \eta^d \right) \Big) \\
&= \frac{1}{\eta} \; \epsilon^{abc} \Big(D_c \omega \; (\cancel{\eta^d D_d \eta_b} - \cancel{\eta^d D_d \eta_b}) \\
&\qquad + \eta_b \; \left(D_c (\eta^d D_d w) - \cancel{D_d w \cdot D_c \eta^d} + \cancel{D_d w \cdot D_c \eta^d} \right) \Big) \\
&= \frac{1}{\eta} \; \epsilon^{abc} \eta_b \; D_c (\cancelto{0}{\mathcal{L}_\eta w}) \\
&= 0 \; ,
\end{aligned} \tag{3.1.24}$$

with the canceling in the third line above, coming from (3.1.14). The solution for S^a therefore satisfies condition (3.1.12a).

Taking the product of S^a with η_a, condition (3.1.12b):

$$S^a \eta_a \;=\; \frac{1}{\eta}\epsilon^{abc}\eta_b D_c\omega\ \eta_a \;=\; \frac{1}{\eta}\epsilon^{abc}\eta_a\eta_b D_c\omega\ . \tag{3.1.25}$$

The alternating tensor ϵ^{abc} is antisymmetric in its indices, and since all of the indices are summed over, any two indices in (3.1.25) can be swapped without effect, hence:

$$\begin{aligned} \epsilon^{abc}\eta_a\eta_b D_c\omega \;&=\; -\,\epsilon^{bac}\eta_b\eta_a D_c\omega \;=\; -\epsilon^{abc}\eta_a\eta_b D_c\omega\ , \\ \Leftrightarrow \qquad \eta_a\eta_b D_c\omega \;&= 0\ , \\ \Leftrightarrow \qquad S^a\eta_a \;&= 0\ , \end{aligned} \tag{3.1.26}$$

satisfying condition (3.1.12b).

The divergence of S^a (3.1.12c) is given directly by:

$$\begin{aligned} D_a S^a \;&= D_a\left(\frac{1}{\eta}\epsilon^{abc}\eta_b D_c\omega\right) \\ &= \cancelto{0}{\frac{1}{\eta}\epsilon^{abc}\eta_b D_a D_c\omega} \;+\; \frac{1}{\eta}\epsilon^{abc}D_c\omega D_a\eta_b \;-\; \frac{1}{\eta^2}\epsilon^{abc}\eta_b D_c\omega D_a\eta\ , \end{aligned} \tag{3.1.27}$$

with the first term canceled due to the antisymmetry of ϵ^{abc}, and the equivalence of mixed partial derivatives:

$$\epsilon^{abc}D_a D_c\ \omega \;=\; -\epsilon^{abc}D_c D_a\ \omega \qquad \Leftrightarrow \qquad D_a D_c\ \omega \;=\; 0\ . \tag{3.1.28}$$

Hence, multiplying both sides of (3.1.27) by η, and by use of (3.1.13b):

$$\begin{aligned} \eta\ D_a S^a \;&= \epsilon^{abc}D_c\omega\ D_a\eta_b \;-\; \epsilon^{abc}\eta_b\ D_c\omega\ D_a\ln\eta \\ &= \frac{1}{2}\epsilon^{abc}D_c\omega\ \eta_b D_a\ln\eta \;-\; \frac{1}{2}\epsilon^{abc}D_c\omega\ \eta_a D_b\ln\eta \;-\; \epsilon^{abc}\eta_b\ D_c\omega\ D_a\ln\eta \\ &= -\,\frac{1}{2}\epsilon^{abc}D_c\omega\left(\eta_a\ D_b\ln\eta + \eta_b\ D_a\ln\eta\right)\ , \end{aligned} \tag{3.1.29}$$

again, due to the antisymmetry of ϵ^{abc}:

$$\begin{aligned} \epsilon^{abc}\left(\eta_a D_b\ln\eta + \eta_b D_a\ln\eta\right) \;&=\; -\,\epsilon^{abc}\left(\eta_b D_a\ln\eta + \eta_a D_b\ln\eta\right)\ , \\ \Leftrightarrow \qquad \eta_a D_b\ln\eta + \eta_b D_a\ln\eta \;&= 0\ , \\ \Leftrightarrow \qquad D_a S^a \;&= 0\ . \end{aligned} \tag{3.1.30}$$

Hence, by equations (3.1.24), (3.1.26) and (3.1.30), all three of the conditions (3.1.12) are satisfied by S^a of (3.1.14).

3.1.6 Comparing *Sergio Dain* with *Baker & Puzio*

Taking the axially symmetric metric, used in *Baker and Puzio* [5], equation (3.1.1):

$$\gamma_{ab}\, dx^a dx^b \;=\; A(x,y)(dx^2 + dy^2) + B(x,y) d\phi^2, \tag{3.1.31}$$

and taking $\eta^a = (0,0,1)$, its covector and norm become:

$$\eta_a \;=\; (0,0,B)\;, \qquad \eta \;=\; \eta^a \eta_a \;=\; B. \tag{3.1.32}$$

The extrinsic curvature from *Dain* [19], (3.1.15), is then only non-zero for terms with either a or b as ϕ, i.e. for $\eta^a = \eta^\phi$ or $\eta^b = \eta^\phi$:

$$\begin{aligned} K^{\phi x} \;=\; K^{x\phi} &= \epsilon^{xjk}\, \frac{\eta_j \eta^\phi}{\eta^2}\, D_k\omega \;=\; \epsilon^{x\phi k}\, \frac{\eta_\phi \eta^\phi}{\eta^2}\, D_k\omega \\ &= -\, \frac{1}{A\sqrt{B}}\, \frac{1}{B}\, D_y\omega \\ &= -\, \frac{1}{AB^{3/2}}\, \partial_y\omega\;, \end{aligned} \tag{3.1.33a}$$

$$\begin{aligned} K^{\phi y} \;=\; K^{y\phi} &= \epsilon^{yjk}\, \frac{\eta_j \eta^\phi}{\eta^2}\, D_k\omega \;=\; \epsilon^{y\phi k}\, \frac{\eta_\phi \eta^\phi}{\eta^2}\, D_k\omega \\ &= \frac{1}{A\sqrt{B}}\, \frac{1}{B}\, D_x\omega \\ &= \frac{1}{AB^{3/2}}\, \partial_x\omega\;, \end{aligned} \tag{3.1.33b}$$

$$\begin{aligned} K^{\phi\phi} &= \epsilon^{\phi jk}\, \frac{\eta_j \eta^\phi}{\eta^2}\, D_k\omega \;+\; \epsilon^{\phi jk}\, \frac{\eta_j \eta^\phi}{\eta^2}\, D_k\omega \\ &= \cancelto{0}{\epsilon^{\phi\phi k}}\, \frac{\eta_\phi \eta^\phi}{\eta^2}\, D_k\omega \;+\; \cancelto{0}{\epsilon^{\phi\phi k}}\, \frac{\eta_\phi \eta^\phi}{\eta^2}\, D_k\omega \\ &= 0\;. \end{aligned} \tag{3.1.33c}$$

We thus get $K^{ab} = 0$, except for the components given by (3.1.33a) and (3.1.33b):

$$\begin{aligned} K^{\phi x} \;&=\; K^{x\phi} \;=\; -\, \frac{1}{AB^{3/2}}\, \partial_y\omega\;, \\ K^{\phi y} \;&=\; K^{y\phi} \;=\; \frac{1}{AB^{3/2}}\, \partial_x\omega\;, \end{aligned} \tag{3.1.34}$$

and lowering the indices, according to $K_{ab} = \gamma_{ai}\gamma_{bj}K^{ij}$:

$$\begin{aligned} K_{\phi x} \;&=\; K_{x\phi} \;=\; -\, B^{-1/2}\, \partial_y\omega\;, \\ K_{\phi y} \;&=\; K_{y\phi} \;=\; B^{-1/2}\, \partial_x\omega\;. \end{aligned} \tag{3.1.35}$$

which agrees exactly with (3.1.5), or equation (2) of *Baker and Puzio* [5].

3.2 Flat Space Scalar Potentials

This section begins by giving the flat 3-space metric in spherical-polar coordinates, and calculating the connection coefficients. The "constraint" equations for a transverse trace-free $(2,0)$-type tensor, see equations (3.0.1), can then be given explicitly, in terms of the tensor components. The axial symmetry condition is applied, and the equations manipulated to give the tensor in terms of 2 independent scalar potentials. This section is mainly based on unpublished work carried out by Prof. Niall Ó Murchadha.

This technique is then repeated for cylindrical coordinates in section 3.2.2, and in section 3.2.3, the scalar potentials for the two coordinate systems are related, through coordinate transformations.

3.2.1 Spherical Coordinates

The flat 3-space metric, in spherical-polar coordinates $(r,\ \theta,\ \phi)$, is given in matrix form, as in equation (1.2.23), by:

$$\gamma_{ab} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} . \qquad (3.2.1)$$

From equation (1.2.36), the connection coefficients for a diagonal metric reduce to:

$$\Gamma^c_{ab} \;=\; \frac{1}{2}\gamma^{cc}\left(\partial_b\gamma_{ac} + \partial_a\gamma_{bc} - \partial_c\gamma_{ab}\right) , \qquad (3.2.2)$$

giving the non-zero connection coefficients, in spherical coordinates, to be:

$$\begin{aligned} \Gamma^r_{\theta\theta} &= -r , & \Gamma^\theta_{r\theta} &= \Gamma^\theta_{\theta r} = \frac{1}{r} , & \Gamma^\phi_{r\phi} &= \Gamma^\phi_{\phi r} = \frac{1}{r} , \\ \Gamma^r_{\phi\phi} &= -r\ \sin^2\theta , & \Gamma^\theta_{\phi\phi} &= -\cos\theta\ \sin\theta , & \Gamma^\phi_{\theta\phi} &= \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta} . \end{aligned} \qquad (3.2.3)$$

The divergence of a tensor T^{ab}, from equation (1.2.34):

$$D_b\, T^{ab} \;=\; \partial_b\, T^{ab} + \Gamma^a_{bc}\, T^{cb} + \Gamma^b_{bc}\, T^{ac} , \qquad (3.2.4)$$

can be given explicitly for the flat space metric, in spherical coordinates (3.2.1), using the connection coefficients (3.2.3). Separating the equations for each value of a:

$$\begin{aligned} D_b T^{rb} &= \partial_b T^{rb} + \Gamma^r_{bc}\, T^{cb} + \Gamma^b_{bc}\, T^{rc} \\ &= \partial_r T^{rr} + \partial_\theta T^{r\theta} + \partial_\phi T^{r\phi} \\ &\quad + \Gamma^r_{\theta\theta}\, T^{\theta\theta} + \Gamma^r_{\phi\phi}\, T^{\phi\phi} + \Gamma^\theta_{\theta r}\, T^{rr} + \Gamma^\phi_{\phi r}\, T^{rr} + \Gamma^\phi_{\phi\theta}\, T^{r\theta} \\ &= \partial_r T^{rr} + \partial_\theta T^{r\theta} + \partial_\phi T^{r\phi} \\ &\quad + \frac{2}{r}\, T^{rr} - r\, T^{\theta\theta} - r\sin^2\theta\, T^{\phi\phi} + \frac{\cos\theta}{\sin\theta}\, T^{r\theta}\ , \end{aligned} \tag{3.2.5a}$$

$$\begin{aligned} D_b T^{\theta b} &= \partial_b T^{\theta b} + \Gamma^\theta_{bc}\, T^{cb} + \Gamma^b_{bc}\, T^{\theta c} \\ &= \partial_r T^{\theta r} + \partial_\theta T^{\theta\theta} + \partial_\phi T^{\theta\phi} \\ &\quad + 2\ \Gamma^\theta_{\theta r}\, T^{r\theta} + \Gamma^\theta_{\phi\phi}\, T^{\phi\phi} + \Gamma^\theta_{\theta r}\, T^{\theta r} + \Gamma^\phi_{\phi r}\, T^{\theta r} + \Gamma^\phi_{\phi\theta}\, T^{\theta\theta} \\ &= \partial_r T^{\theta r} + \partial_\theta T^{\theta\theta} + \partial_\phi T^{\theta\phi} \\ &\quad + \frac{\cos\theta}{\sin\theta}\, T^{\theta\theta} - \cos\theta\ \sin\theta\, T^{\phi\phi} + \frac{4}{r}\, T^{r\theta}\ , \end{aligned} \tag{3.2.5b}$$

$$\begin{aligned} D_b T^{\phi b} &= \partial_b T^{\phi b} + \Gamma^\phi_{bc}\, T^{cb} + \Gamma^b_{bc}\, T^{\phi c} \\ &= \partial_r T^{\phi r} + \partial_\theta T^{\phi\theta} + \partial_\phi T^{\phi\phi} \\ &\quad + 2\ \Gamma^\phi_{\phi r}\, T^{r\phi} + 2\ \Gamma^\phi_{\phi\theta}\, T^{\theta\phi} + \Gamma^\theta_{\theta r}\, T^{\phi r} + \Gamma^\phi_{\phi r}\, T^{\phi r} + \Gamma^\phi_{\phi\theta}\, T^{\phi\theta} \\ &= \partial_r T^{\phi r} + \partial_\theta T^{\phi\theta} + \partial_\phi T^{\phi\phi} + \frac{4}{r}\, T^{r\phi} + 3\ \frac{\cos\theta}{\sin\theta}\, T^{\theta\phi}\ . \end{aligned} \tag{3.2.5c}$$

giving the flat divergence of T^{ab}, completely in terms of the spherical-polar coordinates.

If a symmetric tensor T^{ab} is to be axially symmetric, then its Lie derivative, with respect to an axially-symmetric Killing vector field η^a, must be zero. Hence, by equation (1.2.52) for the Lie derivative of a tensor, and (1.2.34) and (1.2.32) for the *covariant* derivatives of a tensor and vector field respectively:

$$\begin{aligned} 0 &= \mathcal{L}_\eta T^{ab} \\ &= \eta^c \nabla_c T^{ab} - T^{cb}\nabla_c \eta^a - T^{ac}\nabla_c \eta^b \\ &= \eta^c \partial_c T^{ab} + \cancel{\eta^c \Gamma^a_{cd} T^{db}} + \cancel{\eta^c \Gamma^b_{cd} T^{ad}} \\ &\quad - T^{cb}\partial_c \eta^a - \cancel{T^{cb}\Gamma^a_{cd}\eta^d} - T^{ac}\partial_c \eta^b - \cancel{T^{ac}\Gamma^b_{cd}\eta^d}\ , \end{aligned} \tag{3.2.6}$$

with the canceling due to the fact that the indices c and d are completely summed over, the connection coefficients are symmetric in their lower indices, and T^{ab} is symmetric.

Taking the Killing vector to coincide with the azimuthal coordinate vector ϕ, equation (3.2.6) reduces to:

$$0 = \phi^c \partial_c T^{ab} - T^{cb} \cancel{\partial_c \phi^a} - T^{ac} \cancel{\partial_c \phi^b} ,$$

$$\Leftrightarrow \qquad \partial_\phi T^{ab} = 0 , \tag{3.2.7}$$

giving a simple condition for a symmetric tensor T^{ab} to be axially symmetric.

A symmetric tensor T^{ab} can then be assumed to be transverse, trace-free and axially symmetric, by imposing the "constraint" conditions (3.0.1), along with (3.2.7):

$$D_b T^{ab} = 0 , \tag{3.2.8a}$$

$$\gamma_{ab} T^{ab} = 0 , \tag{3.2.8b}$$

$$\partial_\phi T^{ab} = 0 . \tag{3.2.8c}$$

Hence, from the divergence of T^{ab} (3.2.5), the metric (3.2.1), and equation (3.2.8c) above, the transverse and trace-free equations (3.2.8a) and (3.2.8b) become:

$$0 = \partial_r T^{rr} + \partial_\theta T^{r\theta} + \frac{2}{r}\, T^{rr} - r\, T^{\theta\theta} - r \sin^2\theta\, T^{\phi\phi} + \frac{\cos\theta}{\sin\theta}\, T^{r\theta} , \tag{3.2.9a}$$

$$0 = \partial_r T^{\theta r} + \partial_\theta T^{\theta\theta} + \frac{\cos\theta}{\sin\theta}\, T^{\theta\theta} - \cos\theta\, \sin\theta\, T^{\phi\phi} + \frac{4}{r}\, T^{r\theta} , \tag{3.2.9b}$$

$$0 = \partial_r T^{\phi r} + \partial_\theta T^{\phi\theta} + \frac{4}{r}\, T^{r\phi} + 3\frac{\cos\theta}{\sin\theta}\, T^{\theta\phi} , \tag{3.2.9c}$$

$$0 = T^{rr} + r^2\, T^{\theta\theta} + r^2 \sin^2\theta\, T^{\phi\phi} , \tag{3.2.9d}$$

giving a complete set of equations, restricting a symmetric tensor T^{ab} to be transverse, trace-free and axially-symmetric, in a flat 3-space, in spherical-polar coordinates.

Taking equation (3.2.9a), and simplifying with (3.2.9d):

$$
\begin{aligned}
0 &= \partial_r T^{rr} + \partial_\theta T^{r\theta} + \frac{2}{r}\, T^{rr} - r\, T^{\theta\theta} - r \sin^2\theta\, T^{\phi\phi} + \frac{\cos\theta}{\sin\theta}\, T^{r\theta} \\
&= \frac{3}{r}\, T^{rr} + \partial_r T^{rr} + \frac{\cos\theta}{\sin\theta}\, T^{r\theta} + \partial_\theta T^{r\theta} \\
&= \frac{1}{r^3}\, \partial_r \left(r^3\, T^{rr} \right) + \frac{1}{\sin\theta}\, \partial_\theta \left(\sin\theta\, T^{r\theta} \right) \ ,
\end{aligned}
$$

$$
\Leftrightarrow \qquad \partial_r \left(r^3 \sin\theta\, T^{rr} \right) \ = \ \partial_\theta \left(-r^3 \sin\theta\, T^{r\theta} \right) \ , \tag{3.2.10}
$$

hence, by the equivalence of mixed partial derivatives of differential functions, see for example equation (2.1.2), there must exist a scalar potential V such that:

$$
\begin{aligned}
\partial_\theta V &= r^3 \sin\theta\, T^{rr} \ , & \qquad \partial_r V &= - r^3 \sin\theta\, T^{r\theta} \ , \\
\Leftrightarrow \qquad T^{rr} &= \frac{\partial_\theta V}{r^3 \sin\theta} \ , & \qquad T^{r\theta} &= - \frac{\partial_r V}{r^3 \sin\theta} \ .
\end{aligned} \tag{3.2.11}
$$

Now taking equation (3.2.9c):

$$
\begin{aligned}
0 &= \partial_r T^{\phi r} + \partial_\theta T^{\phi\theta} + \frac{4}{r}\, T^{r\phi} + 3\frac{\cos\theta}{\sin\theta}\, T^{\theta\phi} \\
&= \frac{4}{r}\, T^{r\phi} + \partial_r T^{r\phi} + 3\frac{\cos\theta}{\sin\theta}\, T^{\theta\phi} + \partial_\theta T^{\theta\phi} \\
&= \frac{1}{r^4}\, \partial_r \left(r^4\, T^{r\phi} \right) + \frac{1}{\sin^3\theta}\, \partial_\theta \left(\sin^3\theta\, T^{\theta\phi} \right) \ ,
\end{aligned}
$$

$$
\Leftrightarrow \qquad \partial_r \left(r^4 \sin^3\theta\, T^{r\phi} \right) \ = \ \partial_\theta \left(-r^4 \sin^3\theta\, T^{\theta\phi} \right) \ , \tag{3.2.12}
$$

and again, by the equivalence of mixed partial derivatives, there must exist a scalar potential W such that:

$$
\begin{aligned}
\partial_\theta W &= r^4 \sin^3\theta\, T^{r\phi} \ , & \qquad \partial_r W &= - r^4 \sin^3\theta\, T^{\theta\phi} \ , \\
\Leftrightarrow \qquad T^{r\phi} &= \frac{\partial_\theta W}{r^4 \sin^3\theta} \ , & \qquad T^{\theta\phi} &= - \frac{\partial_r W}{r^4 \sin^3\theta} \ .
\end{aligned} \tag{3.2.13}
$$

Finally, using (3.2.9b) with (3.2.9d), gives:

$$\begin{aligned} 0 &= \partial_r T^{\theta r} + \partial_\theta T^{\theta\theta} + \frac{\cos\theta}{\sin\theta}\, T^{\theta\theta} - \cos\theta\ \sin\theta\ T^{\phi\phi} + \frac{4}{r}\, T^{r\theta} \\ &= \partial_r T^{r\theta} + \partial_\theta T^{\theta\theta} + \frac{\cos\theta}{\sin\theta}\, T^{\theta\theta} + \frac{\cos\theta}{r^2\sin\theta}\ \left(T^{rr} + r^2 T^{\theta\theta}\right) + \frac{4}{r}\, T^{r\theta} \\ &= 2\frac{\cos\theta}{\sin\theta}\, T^{\theta\theta} + \partial_\theta T^{\theta\theta} + \frac{4}{r}\, T^{r\theta} + \partial_r T^{r\theta} + \frac{\cos\theta}{r^2\sin\theta}\, T^{rr}\ , \end{aligned}$$

$$\Leftrightarrow \qquad \frac{1}{\sin^2\theta}\,\partial_\theta\left(\sin^2\theta\ T^{\theta\theta}\right) = -\frac{1}{r^4}\,\partial_r\left(r^4\ T^{r\theta}\right) - \frac{\cos\theta}{r^5\sin^2\theta}\,\partial_\theta V\ ,$$

$$\Leftrightarrow \qquad \partial_\theta\left(\sin^2\theta\ T^{\theta\theta}\right) = \frac{\sin\theta}{r^4}\,\partial_r\left(r\ \partial_r V\right) - \frac{\cos\theta}{r^5}\,\partial_\theta V\ ,$$

$$\Leftrightarrow \qquad T^{\theta\theta} = \frac{1}{r^3\sin^2\theta}\int\left[\sin\theta\ \partial_{rr}V + \frac{\sin\theta}{r}\,\partial_r V - \frac{\cos\theta}{r^2}\,\partial_\theta V\right]d\theta\ . \qquad (3.2.14)$$

Note that the addition of a constant of integration can also be included for $T^{\theta\theta}$. This “constant” can depend on r, but not θ (or ϕ).

Hence, transverse, trace-free and axially-symmetric tensors in flat space are given, in spherical coordinates, by:

$$T^{ab} = \begin{pmatrix} \frac{1}{r^3\sin\theta}\,\partial_\theta V & -\frac{1}{r^3\sin\theta}\,\partial_r V & \frac{1}{r^4\sin^3\theta}\,\partial_\theta W \\ -\frac{1}{r^3\sin\theta}\,\partial_r V & \begin{array}{c}\frac{1}{r^3\sin^2\theta}\int\left[\sin\theta\ \partial_{rr}V\right. \\ \left. +\frac{\sin\theta}{r}\,\partial_r V - \frac{\cos\theta}{r^2}\,\partial_\theta V\right]d\theta\end{array} & -\frac{1}{r^4\sin^3\theta}\,\partial_r W \\ \frac{1}{r^4\sin^3\theta}\,\partial_\theta W & -\frac{1}{r^4\sin^3\theta}\,\partial_r W & \begin{array}{c} -\frac{1}{r^5\sin^3\theta}\,\partial_\theta V \\ -\frac{1}{r^5\sin^4\theta}\int\left[\sin\theta\ \partial_{rr}V\right. \\ \left. +\frac{\sin\theta}{r}\,\partial_r V - \frac{\cos\theta}{r^2}\,\partial_\theta V\right]d\theta\end{array} \end{pmatrix}, \qquad (3.2.15)$$

with two scalar potentials V and W.

Again, the addition of a constant of integration (dependent on r but not θ or ϕ) can be included for $T^{\theta\theta}$ and $T^{\phi\phi}$, however, since T^{ab} is trace-free, see equation (3.2.9d), the choice of one constant must determine the other.

3.2.2 Cylindrical Coordinates

In this thesis, cylindrical coordinates are given in the order $(\rho,\ z,\ \phi)$, for easy comparison with the spherical coordinates $(r,\ \theta,\ \phi)$. However it must be noted, that this order produces a "left hand orthogonality", reversing the order for the Levi-Civita tensor (2.2.57). The flat space 3-metric in these cylindrical coordinates, is then given by:

$$\gamma_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho^2 \end{pmatrix} , \tag{3.2.16}$$

with the non-zero flat connection coefficients, in cylindrical coordinates:

$$\Gamma^{\rho}_{\phi\phi} = -\rho \ , \qquad \Gamma^{\phi}_{\rho\phi} = \Gamma^{\phi}_{\phi\rho} = \frac{1}{\rho} \ . \tag{3.2.17}$$

Again, the divergence of a tensor T^{ab}:

$$D_b\, T^{ab} = \partial_b\, T^{ab} + \Gamma^{a}_{bc}\, T^{cb} + \Gamma^{b}_{bc}\, T^{ac} \ , \tag{3.2.18}$$

can be given explicitly for the flat space metric in cylindrical coordinates (3.2.16), using the connection coefficients (3.2.17). Separating the equations for each value of a:

$$\begin{aligned} D_b T^{\rho b} &= \partial_b T^{\rho b} + \Gamma^{\rho}_{bc}\, T^{cb} + \Gamma^{b}_{bc}\, T^{\rho c} \\ &= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} + \partial_\phi T^{\rho\phi} + \Gamma^{\rho}_{\phi\phi}\, T^{\phi\phi} + \Gamma^{\phi}_{\phi\rho}\, T^{\rho\rho} \\ &= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} + \partial_\phi T^{\rho\phi} - \rho\, T^{\phi\phi} + \frac{1}{\rho}\, T^{\rho\rho} \ , \end{aligned} \tag{3.2.19a}$$

$$\begin{aligned} D_b T^{z b} &= \partial_b T^{z b} + \Gamma^{z}_{bc}\, T^{cb} + \Gamma^{b}_{bc}\, T^{z c} \\ &= \partial_\rho T^{z\rho} + \partial_z T^{zz} + \partial_\phi T^{z\phi} + \cancel{\Gamma^{z}_{bc}}\, T^{cb} + \Gamma^{\phi}_{\phi\rho}\, T^{z\rho} \\ &= \partial_\rho T^{z\rho} + \partial_z T^{zz} + \partial_\phi T^{z\phi} + \frac{1}{\rho}\, T^{z\rho} \ , \end{aligned} \tag{3.2.19b}$$

$$\begin{aligned} D_b T^{\phi b} &= \partial_b T^{\phi b} + \Gamma^{\phi}_{bc}\, T^{cb} + \Gamma^{b}_{bc}\, T^{\phi c} \\ &= \partial_\rho T^{\phi\rho} + \partial_z T^{\phi z} + \partial_\phi T^{\phi\phi} + 2\Gamma^{\phi}_{\phi\rho}\, T^{\rho\phi} + \Gamma^{\phi}_{\phi\rho}\, T^{\phi\rho} \\ &= \partial_\rho T^{\phi\rho} + \partial_z T^{\phi z} + \partial_\phi T^{\phi\phi} + \frac{3}{\rho}\, T^{\rho\phi} \ . \end{aligned} \tag{3.2.19c}$$

Recalling the "constraint" equations (3.0.1) for a tensor to be transverse and trace-free, and the condition(3.2.7) for axial symmetry, T^{ab} must satisfy:

$$D_b T^{ab} = 0 \ , \tag{3.2.20a}$$

$$\gamma_{ab} T^{ab} = 0 \ , \tag{3.2.20b}$$

$$\partial_\phi T^{ab} = 0 \ , \tag{3.2.20c}$$

then, by use of the divergence of T^{ab} (3.2.19), the metric (3.2.16), and equation (3.2.20c) above, the transverse and trace-free equations (3.2.20a) and (3.2.20b) for cylindrical coordinates, are given by:

$$
\begin{aligned}
0 &= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} - \rho\, T^{\phi\phi} + \frac{1}{\rho}\, T^{\rho\rho} \;, && (3.2.21a)\\
0 &= \partial_\rho T^{z\rho} + \partial_z T^{zz} + \frac{1}{\rho}\, T^{z\rho} \;, && (3.2.21b)\\
0 &= \partial_\rho T^{\phi\rho} + \partial_z T^{\phi z} + \frac{3}{\rho}\, T^{\rho\phi} \;, && (3.2.21c)\\
0 &= T^{\rho\rho} + T^{zz} + \rho^2\, T^{\phi\phi} \;. && (3.2.21d)
\end{aligned}
$$

Taking equation (3.2.21b):

$$
\begin{aligned}
0 &= \frac{1}{\rho}\, T^{z\rho} + \partial_\rho T^{z\rho} + \partial_z T^{zz}\\
&= \frac{1}{\rho}\, \partial_\rho \left(\rho\, T^{z\rho}\right) + \partial_z \left(T^{zz}\right)\\
&= \frac{1}{\rho}\, \partial_\rho \left(\rho\, T^{z\rho}\right) + \frac{1}{\rho}\, \partial_z \left(\rho\, T^{zz}\right) \;,
\end{aligned}
$$

$$
\Leftrightarrow \qquad \partial_\rho \left(\rho\, T^{z\rho}\right) \;=\; \partial_z \left(-\rho\, T^{zz}\right) \;. \qquad (3.2.22)
$$

Hence, by the equivalence of mixed partial derivatives, there must exist a scalar potential X, such that:

$$
\begin{aligned}
&\partial_z X = \rho\, T^{\rho z} \;, \qquad \partial_\rho X = -\,\rho\, T^{zz} \;,\\
\Leftrightarrow \qquad &T^{\rho z} = \frac{1}{\rho}\partial_z X \;, \qquad T^{zz} = -\,\frac{1}{\rho}\partial_\rho X \;.
\end{aligned} \qquad (3.2.23)
$$

Now taking equation (3.2.21c):

$$
\begin{aligned}
0 &= \frac{3}{\rho}\, T^{\rho\phi} + \partial_\rho T^{\rho\phi} + \partial_z T^{z\phi}\\
&= \frac{1}{\rho^3}\, \partial_\rho \left(\rho^3\, T^{\rho\phi}\right) + \partial_z \left(T^{z\phi}\right)\\
&= \frac{1}{\rho^3}\, \partial_\rho \left(\rho^3\, T^{\rho\phi}\right) + \frac{1}{\rho^3}\, \partial_z \left(\rho^3\, T^{z\phi}\right) \;,
\end{aligned}
$$

$$
\Leftrightarrow \qquad \partial_\rho \left(\rho^3\, T^{\rho\phi}\right) \;=\; \partial_z \left(-\rho^3\, T^{z\phi}\right) \;, \qquad (3.2.24)
$$

and again, by mixed partial derivatives there must exist a scalar potential Y, such that:

$$
\begin{aligned}
&\partial_z Y = \rho^3\, T^{\rho\phi} \;, \qquad \partial_\rho Y = -\,\rho^3\, T^{z\phi} \;,\\
\Leftrightarrow \qquad &T^{\rho\phi} = \frac{1}{\rho^3}\, \partial_z Y \;, \qquad T^{z\phi} = -\,\frac{1}{\rho^3}\, \partial_\rho Y \;.
\end{aligned} \qquad (3.2.25)
$$

Finally, using (3.2.21a) with (3.2.21d):

$$\begin{aligned} 0 &= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} - \rho\, T^{\phi\phi} + \frac{1}{\rho}\, T^{\rho\rho} \\ &= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} + \frac{1}{\rho}\, \left(T^{\rho\rho} + T^{zz}\right) + \frac{1}{\rho}\, T^{\rho\rho} \\ &= \frac{2}{\rho}\, T^{\rho\rho} + \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} + \frac{1}{\rho}\, T^{zz} \\ &= \frac{1}{\rho^2}\, \partial_\rho \left(\rho^2\, T^{\rho\rho}\right) + \frac{1}{\rho}\, \partial_{zz} X - \frac{1}{\rho^2}\, \partial_\rho X \ , \end{aligned}$$

$$\Leftrightarrow \qquad \partial_\rho \left(\rho^2\, T^{\rho\rho}\right) \ = \ \partial_\rho X - \rho\, \partial_{zz} X \ ,$$

$$\Leftrightarrow \qquad T^{\rho\rho} \ = \ \frac{1}{\rho^2} \int \left[\partial_\rho X - \rho\, \partial_{zz} X\right] d\rho \ . \tag{3.2.26}$$

Note that the addition of a constant of integration, as with equation (3.2.14), can also be included for $T^{\rho\rho}$. This "constant" can depend on z, but not ρ (or ϕ).

Hence, transverse, trace-free and axially-symmetric tensors in flat space are given, in cylindrical coordinates, by:

$$T^{ab} = \begin{pmatrix} \frac{1}{\rho^2} \int \left[-\rho\, \partial_{zz} X + \partial_\rho X\right] d\rho & \frac{1}{\rho}\, \partial_z X & \frac{1}{\rho^3} \partial_z Y \\ \frac{1}{\rho}\, \partial_z X & -\frac{1}{\rho}\, \partial_\rho X & -\frac{1}{\rho^3} \partial_\rho Y \\ \frac{1}{\rho^3} \partial_z Y & -\frac{1}{\rho^3} \partial_\rho Y & \frac{1}{\rho^3}\, \partial_\rho X - \frac{1}{\rho^4} \int \left[-\rho\, \partial_{zz} X + \partial_\rho X\right] d\rho \end{pmatrix}, \tag{3.2.27}$$

with two scalar potentials X and Y.

Again, the addition of a constant of integration (dependent on z but not ρ or ϕ) can be included for $T^{\rho\rho}$ and $T^{\phi\phi}$, however, as with (3.2.15), since T^{ab} is trace-free, the choice of one constant must determine the other.

3.2.3 Relating the Coordinate Potentials

A coordinate transformation on T^{ab}, from cylindrical to spherical coordinates, is carried out to find the relationship between the scalar potentials for each. The relations between the spherical and cylindrical coordinates are given by:

$$\begin{aligned} r &= \sqrt{\rho^2 + z^2} \ , & \theta &= ArcTan\left(\frac{\rho}{z}\right) \ , & \phi &= \phi \ , \\ \rho &= r \ \sin\theta \ , & z &= r \ \cos\theta \ , \end{aligned} \tag{3.2.28}$$

and the necessary derivatives of the spherical coordinates with respect to the cylindrical:

$$\begin{aligned} \frac{\partial r}{\partial \rho} &= \frac{\rho}{r} = \sin\theta \ , & \frac{\partial r}{\partial z} &= \frac{z}{r} = \cos\theta \ , \\ \frac{\partial \theta}{\partial \rho} &= \frac{z}{r^2} = \frac{1}{r} \ \cos\theta \ , & \frac{\partial \theta}{\partial z} &= -\frac{\rho}{r^2} = -\frac{1}{r} \ \sin\theta \ . \end{aligned} \tag{3.2.29}$$

The derivatives of the cylindrical potentials X and Y are first found with respect to r and θ, in terms of their derivatives with respect to ρ and z, using the chain rule:

$$\partial_\rho X = \frac{\partial X}{\partial x^i}\frac{\partial x^i}{\partial \rho} = \frac{\partial X}{\partial r}\frac{\partial r}{\partial \rho} + \frac{\partial X}{\partial \theta}\frac{\partial \theta}{\partial \rho} = \sin\theta \ \partial_r X + \frac{1}{r} \ \cos\theta \ \partial_\theta X \ , \tag{3.2.30}$$

$$\partial_z X = \frac{\partial X}{\partial x^i}\frac{\partial x^i}{\partial z} = \frac{\partial X}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial X}{\partial \theta}\frac{\partial \theta}{\partial z} = \cos\theta \ \partial_r X - \frac{1}{r} \ \sin\theta \ \partial_\theta X \ , \tag{3.2.31}$$

$$\partial_\rho Y = \frac{\partial Y}{\partial x^i}\frac{\partial x^i}{\partial \rho} = \frac{\partial Y}{\partial r}\frac{\partial r}{\partial \rho} + \frac{\partial Y}{\partial \theta}\frac{\partial \theta}{\partial \rho} = \sin\theta \ \partial_r Y + \frac{1}{r} \ \cos\theta \ \partial_\theta Y \ , \tag{3.2.32}$$

$$\partial_z Y = \frac{\partial Y}{\partial x^i}\frac{\partial x^i}{\partial z} = \frac{\partial Y}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial Y}{\partial \theta}\frac{\partial \theta}{\partial z} = \cos\theta \ \partial_r Y - \frac{1}{r} \ \sin\theta \ \partial_\theta Y \ , \tag{3.2.33}$$

and since the only second derivative in the cylindrical potentials is $\partial_{zz}X$, this is similarly calculated to be:

$$\begin{aligned} \partial_{zz} X = {} & \cos^2\theta \ \partial_{rr} X + r^{-2}\sin^2\theta \ \partial_{\theta\theta} X - r^{-1} 2\cos\theta \ \sin\theta \ \partial_{r\theta} X \\ & + r^{-1}\sin^2\theta \ \partial_r X + 2 \ r^{-2}\cos\theta \ \sin\theta \ \partial_\theta X \ . \end{aligned} \tag{3.2.34}$$

The tensor components T^{ab} are then transformed from cylindrical to spherical coordinates, according to equation (1.2.56):

$$T^{cd}_{sph} = \frac{\partial x^c_{sph}}{\partial x^a_{cyl}}\frac{\partial x^d_{sph}}{\partial x^b_{cyl}} \ T^{ab}_{cyl} \ , \tag{3.2.35}$$

giving the spherical components in terms of X and Y, in the following equations.

$$\begin{aligned}
T^{rr} &= \frac{\partial r}{\partial x^a}\frac{\partial r}{\partial x^b}\,T^{ab} = \frac{\partial r}{\partial \rho}\frac{\partial r}{\partial \rho}\,T^{\rho\rho} + \frac{\partial r}{\partial \rho}\frac{\partial r}{\partial z}\,T^{\rho z} + \frac{\partial r}{\partial z}\frac{\partial r}{\partial \rho}\,T^{z\rho} + \frac{\partial r}{\partial z}\frac{\partial r}{\partial z}\,T^{zz} \\
&= \sin^2\theta\,\frac{1}{\rho^2}\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho + 2\,\sin\theta\,\cos\theta\,\frac{1}{\rho}\,\partial_z X + \cos^2\theta\,\frac{-1}{\rho}\,\partial_\rho X \\
&= \frac{1}{r^2}\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho + \frac{2\cos\theta}{r}\left(\cos\theta\,\partial_r X - \frac{1}{r}\,\sin\theta\,\partial_\theta X\right) \\
&\quad - \frac{\cos^2\theta}{r\,\sin\theta}\left(\sin\theta\,\partial_r X + \frac{1}{r}\,\cos\theta\,\partial_\theta X\right) \\
&= \frac{1}{r^2}\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho + \frac{\cos^2\theta}{r}\,\partial_r X - \frac{2\,\cos\theta\,\sin\theta}{r^2}\,\partial_\theta X - \frac{\cos^3\theta}{r^2\,\sin\theta}\,\partial_\theta X\ ,
\end{aligned} \tag{3.2.36}$$

$$\begin{aligned}
T^{\theta\theta} &= \frac{\partial \theta}{\partial x^a}\frac{\partial \theta}{\partial x^b}\,T^{ab} = \frac{\partial \theta}{\partial \rho}\frac{\partial \theta}{\partial \rho}\,T^{\rho\rho} + \frac{\partial \theta}{\partial \rho}\frac{\partial \theta}{\partial z}\,T^{\rho z} + \frac{\partial \theta}{\partial z}\frac{\partial \theta}{\partial \rho}\,T^{z\rho} + \frac{\partial \theta}{\partial z}\frac{\partial \theta}{\partial z}\,T^{zz} \\
&= \frac{\cos^2\theta}{r^2}\,\frac{1}{\rho^2}\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho - 2\,\frac{\sin\theta\,\cos\theta}{r^2}\,\frac{1}{\rho}\,\partial_z X + \frac{\sin^2\theta}{r^2}\,\frac{-1}{\rho}\,\partial_\rho X \\
&= \frac{\cos^2\theta}{r^4\sin^2\theta}\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho - \frac{2\,\cos\theta}{r^3}\left(\cos\theta\,\partial_r X - \frac{1}{r}\,\sin\theta\,\partial_\theta X\right) \\
&\quad - \frac{\sin\theta}{r^3}\left(\sin\theta\,\partial_r X + \frac{1}{r}\,\cos\theta\,\partial_\theta X\right) \\
&= \frac{\cos^2\theta}{r^4\sin^2\theta}\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho - \frac{(1+\cos^2\theta)}{r^3}\,\partial_r X + \frac{\cos\theta\,\sin\theta}{r^4}\,\partial_\theta X\ ,
\end{aligned} \tag{3.2.37}$$

$$\begin{aligned}
T^{\phi\phi} &= \frac{\partial \phi}{\partial x^a}\frac{\partial \phi}{\partial x^b}\,T^{ab} = \frac{\partial \phi}{\partial \phi}\frac{\partial \phi}{\partial \phi}\,T^{\phi\phi} = \frac{1}{\rho^3}\,\partial_\rho X - \frac{1}{\rho^4}\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho \\
&= \frac{1}{r^3\sin^3\theta}\,\partial_\rho X - \frac{1}{r^4\sin^4\theta}\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho \\
&= \frac{1}{r^3\sin^3\theta}\left(\sin\theta\,\partial_r X + \frac{1}{r}\,\cos\theta\,\partial_\theta X\right) - \frac{1}{r^4\sin^4\theta}\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho \\
&= \frac{1}{r^3\sin^2\theta}\,\partial_r X + \frac{\cos\theta}{r^4\sin^3\theta}\,\partial_\theta X - \frac{1}{r^4\sin^4\theta}\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho\ ,
\end{aligned} \tag{3.2.38}$$

$$
\begin{aligned}
T^{r\theta} &= \frac{\partial r}{\partial x^a}\frac{\partial \theta}{\partial x^b}\,T^{ab} = \frac{\partial r}{\partial \rho}\frac{\partial \theta}{\partial \rho}\,T^{\rho\rho} + \frac{\partial r}{\partial \rho}\frac{\partial \theta}{\partial z}\,T^{\rho z} + \frac{\partial r}{\partial z}\frac{\partial \theta}{\partial \rho}\,T^{z\rho} + \frac{\partial r}{\partial z}\frac{\partial \theta}{\partial z}\,T^{zz} \\
&= \sin\theta\frac{\cos\theta}{r}\,\frac{1}{\rho^2}\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho + \sin\theta\frac{-\sin\theta}{r}\frac{1}{\rho}\partial_z X \\
&\quad + \cos\theta\frac{\cos\theta}{r}\,\frac{1}{\rho}\partial_z X + \cos\theta\frac{-\sin\theta}{r}\,\frac{-1}{\rho}\partial_\rho X \\
&= \frac{\cos\theta}{r^3\sin\theta}\,\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho - \frac{\sin\theta}{r^2}\,\left(\cos\theta\,\partial_r X - \frac{\sin\theta}{r}\,\partial_\theta X\right) \\
&\quad + \frac{\cos^2\theta}{r^2\sin\theta}\,\left(\cos\theta\,\partial_r X - \frac{\sin\theta}{r}\,\partial_\theta X\right) + \frac{\cos\theta}{r^2}\,\left(\sin\theta\,\partial_r X + \frac{\cos\theta}{r}\,\partial_\theta X\right) \\
&= \frac{\cos\theta}{r^3\sin\theta}\,\int\left[-\rho\,\partial_{zz}X + \partial_\rho X\right]d\rho + \frac{\sin^2\theta}{r^3}\,\partial_\theta X + \frac{\cos^3\theta}{r^2\sin\theta}\,\partial_r X\ , \qquad (3.2.39)
\end{aligned}
$$

$$
\begin{aligned}
T^{r\phi} &= \frac{\partial r}{\partial x^i}\frac{\partial \phi}{\partial \phi}\,T^{i\phi} = \frac{\partial r}{\partial \rho}\,T^{\rho\phi} + \frac{\partial r}{\partial z}\,T^{z\phi} = \sin\theta\,\frac{1}{\rho^3}\partial_z Y - \cos\theta\,\frac{1}{\rho^3}\partial_\rho Y \\
&= \frac{\sin\theta}{r^3\sin^3\theta}\left(\cos\theta\,\partial_r Y - \frac{\sin\theta}{r}\,\partial_\theta Y\right) - \frac{\cos\theta}{r^3\sin^3\theta}\left(\sin\theta\,\partial_r Y + \frac{\cos\theta}{r}\,\partial_\theta Y\right) \\
&= \frac{1}{r^3\sin^3\theta}\left(\left(\cancel{\cos\theta\,\sin\theta} - \cancel{\cos\theta\,\sin\theta}\right)\,\partial_r Y - \left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right)\,\partial_\theta Y\right) \\
&= -\,\frac{\partial_\theta Y}{r^4\sin^3\theta}\ , \qquad (3.2.40)
\end{aligned}
$$

$$
\begin{aligned}
T^{\theta\phi} &= \frac{\partial \theta}{\partial x^i}\frac{\partial \phi}{\partial \phi}\,T^{i\phi} = \frac{\partial \theta}{\partial \rho}\,T^{\rho\phi} + \frac{\partial \theta}{\partial z}\,T^{z\phi} = \frac{\cos\theta}{r}\,\frac{1}{\rho^3}\partial_z Y + \frac{\sin\theta}{r}\,\frac{1}{\rho^3}\partial_\rho Y \\
&= \frac{\cos\theta}{r^4\sin^3\theta}\,\left(\cos\theta\,\partial_r Y - \frac{\sin\theta}{r}\,\partial_\theta Y\right) + \frac{\sin\theta}{r^4\sin^3\theta}\,\left(\sin\theta\,\partial_r Y + \frac{\cos\theta}{r}\,\partial_\theta Y\right) \\
&= \frac{1}{r^4\sin^3\theta}\,\left(\left(\cancel{\cos^2\theta + \sin^2\theta}\right)\,\partial_r Y + \left(\cancel{\frac{\cos\theta\,\sin\theta}{r}} - \cancel{\frac{\cos\theta\,\sin\theta}{r}}\right)\,\partial_\theta Y\right) \\
&= \frac{\partial_r Y}{r^4\sin^3\theta}\ . \qquad (3.2.41)
\end{aligned}
$$

Substituting the spherical components of the the transverse trace-free tensors from (3.2.15) into (3.2.36) and rearranging:

$$\begin{aligned}\frac{1}{r^3 \sin\theta}\ \partial_\theta V\ &= \frac{1}{r^2}\int \left[-\rho\ \partial_{zz}X + \partial_\rho X\right] d\rho \\ &+ \frac{\cos^2\theta}{r}\ \partial_r X - \frac{2\ \cos\theta\ \sin\theta}{r^2}\ \partial_\theta X - \frac{\cos^3\theta}{r^2\ \sin\theta}\ \partial_\theta X\ ,\end{aligned}$$

$$\begin{aligned}\Leftrightarrow\qquad \int \left[-\rho\ \partial_{zz}X + \partial_\rho X\right] d\rho = \frac{1}{r\sin\theta}\ \partial_\theta V\ - r\ \cos^2\theta\ \partial_r X \\ + 2\ \cos\theta\ \sin\theta\ \partial_\theta X + \frac{\cos^3\theta}{\sin\theta}\ \partial_\theta X\ .\end{aligned} \tag{3.2.42}$$

Now substituting from (3.2.15), into (3.2.39):

$$-\frac{1}{r^3\sin\theta}\ \partial_r V = \frac{\cos\theta}{r^3\sin\theta}\ \int \left[-\rho\ \partial_{zz}X + \partial_\rho X\right] d\rho + \frac{\sin^2\theta}{r^3}\ \partial_\theta X + \frac{\cos^3\theta}{r^2\sin\theta}\ \partial_r X\ ,$$

$$\Leftrightarrow\qquad \int \left[-\rho\ \partial_{zz}X + \partial_\rho X\right] d\rho = -\frac{1}{\cos\theta}\ \partial_r V + \frac{\sin^3\theta}{\cos\theta}\ \partial_\theta X + r\ \cos^2\theta\ \partial_r X\ , \tag{3.2.43}$$

and equating the right-hand sides of (3.2.42) and (3.2.43):

$$\begin{aligned}&\frac{1}{r\sin\theta}\ \partial_\theta V\ - r\ \cos^2\theta\ \partial_r X + 2\ \cos\theta\ \sin\theta\ \partial_\theta X + \frac{\cos^3\theta}{\sin\theta}\ \partial_\theta X \\ =\ &-\frac{1}{\cos\theta}\ \partial_r V + \frac{\sin^3\theta}{\cos\theta}\ \partial_\theta X + r\ \cos^2\theta\ \partial_r X\ ,\end{aligned}$$

$$\begin{aligned}\Leftrightarrow\qquad &\cos\theta\ \partial_\theta V\ - r\ \sin\theta\ \partial_r V \\ &= +r\ \sin^4\theta\ \partial_\theta X + 2\ r\ \cos^2\theta\ \sin^2\theta\ \partial_\theta X + r\ \cos^4\theta\ \partial_\theta X\ ,\end{aligned}$$

$$\Leftrightarrow\qquad \cos\theta\ \partial_\theta V\ - r\ \sin\theta\ \partial_r V = \ r\ \partial_\theta X\ ,$$

$$\Leftrightarrow\qquad X = \int \left[\frac{1}{r}\ \cos\theta\ \partial_\theta V\ - \sin\theta\ \partial_r V\right] d\theta\ , \tag{3.2.44}$$

giving the cylindrical potential X in terms of the spherical potential V.

Substituting from (3.2.15) into (3.2.37) and rearranging:

$$\frac{1}{r^3 \sin^2\theta} \int \left[\sin\theta \; \partial_{rr} V + \frac{\sin\theta}{r} \; \partial_r V - \frac{\cos\theta}{r^2} \; \partial_\theta V \right] d\theta$$
$$= \frac{\cos^2\theta}{r^4 \sin^2\theta} \; \int \left[-\rho \; \partial_{zz} X + \partial_\rho X \right] d\rho - \frac{(1+\cos^2\theta)}{r^3} \; \partial_r X + \frac{\cos\theta \; \sin\theta}{r^4} \; \partial_\theta X \; ,$$

$$\Leftrightarrow \qquad \int \left[\sin\theta \; \partial_{rr} V + \frac{\sin\theta}{r} \; \partial_r V - \frac{\cos\theta}{r^2} \; \partial_\theta V \right] d\theta$$
$$= \frac{\cos^2\theta}{r} \; \int \left[-\rho \; \partial_{zz} X + \partial_\rho X \right] d\rho - (1+\cos^2\theta) \; \sin^2\theta \; \partial_r X + \frac{\cos\theta \; \sin^3\theta}{r} \; \partial_\theta X \; , \tag{3.2.45}$$

and substituting from (3.2.15) into (3.2.38):

$$-\frac{1}{r^5 \sin^3\theta} \; \partial_\theta V - \frac{1}{r^5 \sin^4\theta} \int \left[\sin\theta \; \partial_{rr} V + \frac{\sin\theta}{r} \; \partial_r V - \frac{\cos\theta}{r^2} \; \partial_\theta V \right] d\theta$$
$$= \frac{1}{r^3 \sin^2\theta} \; \partial_r X + \frac{\cos\theta}{r^4 \sin^3\theta} \; \partial_\theta X - \frac{1}{r^4 \sin^4\theta} \int \left[-\rho \; \partial_{zz} X + \partial_\rho X \right] d\rho \; ,$$

$$\Leftrightarrow \qquad \int \left[\sin\theta \; \partial_{rr} V + \frac{\sin\theta}{r} \; \partial_r V - \frac{\cos\theta}{r^2} \; \partial_\theta V \right] d\theta$$
$$= r \int \left[-\rho \; \partial_{zz} X + \partial_\rho X \right] d\rho - \sin\theta \; \partial_\theta V - r^2 \sin^2\theta \; \partial_r X - r \; \cos\theta \; \sin\theta \; \partial_\theta X \; , \tag{3.2.46}$$

Equating the right-hand sides of (3.2.45) and (3.2.46):

$$\frac{\cos^2\theta}{r} \; \int \left[-\rho \; \partial_{zz} X + \partial_\rho X \right] d\rho - (1+\cos^2\theta) \; \sin^2\theta \; \partial_r X + \frac{\cos\theta \; \sin^3\theta}{r} \; \partial_\theta X$$
$$= r \int \left[-\rho \; \partial_{zz} X + \partial_\rho X \right] d\rho - \sin\theta \; \partial_\theta V - r^2 \sin^2\theta \; \partial_r X - r \; \cos\theta \; \sin\theta \; \partial_\theta X \; ,$$

$$\Leftrightarrow \qquad r \; \sin\theta \; \partial_\theta V \; = (r^2 - \cos^2\theta) \int \left[-\rho \; \partial_{zz} X + \partial_\rho X \right] d\rho$$
$$+ r \sin^2\theta (1 - r^2 + \cos^2\theta) \; \partial_r X - \cos\theta \; \sin\theta (r^2 + \sin^2\theta) \partial_\theta X \; ,$$

$$\Leftrightarrow \qquad V \; = \int \left[\frac{r^2 - \cos^2\theta}{r \; \sin\theta} \int \left[-\rho \; \partial_{zz} X + \partial_\rho X \right] d\rho \right.$$
$$\left. + \sin\theta (1 - r^2 + \cos^2\theta) \; \partial_r X - \frac{1}{r} \; \cos\theta (r^2 + \sin^2\theta) \partial_\theta X \right] d\theta \; , \tag{3.2.47}$$

giving the spherical potential V in terms of the cylindrical potential X, though with a double integral.

Finally, substituting from (3.2.15) into (3.2.40) and (3.2.41):

$$\frac{1}{r^4 \sin^3\theta}\,\partial_\theta W = -\frac{\partial_\theta Y}{r^4 \sin^3\theta}\ , \qquad -\frac{1}{r^4 \sin^3\theta}\,\partial_r W = \frac{\partial_r Y}{r^4 \sin^3\theta}\ ,$$

$$\Leftrightarrow \qquad \partial_\theta W \;=\; -\partial_\theta Y\ , \qquad -\,\partial_r W \;=\; \partial_r Y\ ,$$

$$\Leftrightarrow \qquad W \;=\; -Y + \text{constant}\ , \tag{3.2.48}$$

giving an equivalence, up to sign, for the spherical potential W and the cylindrical potential Y. The constant arises from the integration, and since the integral is carried out with respect to both r and θ, the constant cannot have a dependence on either. The constant also means little, since each potential is differentiated by r or θ, or their linear transformations ρ or z, before being used to form a transverse trace-free tensor.

For the potentials V and X, the relations (3.2.44) and (3.2.47) are not quite as effective as (3.2.48) for W and Y. In order to test the relations, a number of choices were made for X in cylindrical coordinates. *Mathematica* was then used to transform the resulting TT tensor into spherical coordinates, and to find the potential V.

For potentials given by multiplying different powers of ρ and z together, the following relations were found for $X \to V$:

$$\begin{array}{rclrclrcl}
 & & & z & \to & \frac{1}{2}\rho^2 - \frac{1}{2}z^2\ , & z^2 & \to & \frac{2}{2}z\rho^2 - \frac{2}{3}z^3\ , \\
\rho & \to & -z\rho\ , & z\rho & \to & \frac{1}{3}\rho^3 - z^2\rho\ , & z^2\rho & \to & \frac{2}{3}z\rho^3 - z^3\rho\ , \\
\rho^2 & \to & -z\rho^2\ , & z\rho^2 & \to & \frac{1}{4}\rho^4 - z^2\rho^2\ , & z^2\rho^2 & \to & \frac{2}{4}z\rho^4 - z^3\rho^2\ .
\end{array} \tag{3.2.49}$$

Note that there will always exist an extra constant of integration to each of the terms above, however, since this constant will be differentiated out before use in a TT tensor, it has been omitted in the results above.

Though the equations (3.2.44) and (3.2.47) were found to agree with the *Mathematica* calculations, it was also hoped that a trend could be found from the results (3.2.49). Unfortunately, the transformations in (3.2.49) show signs that the integrals play a dominant role in the transformations, and hence, a more simple expression is unlikely.

As a result of the difficulty involved in relating the two potentials V and X, it has not been possible to derive a coordinate-free expression for the axially-symmetric, transverse trace-free tensors in flat space, similar to *Dain* [19], in equation (3.1.15), though with further study, this may still be possible.

3.3 General Space Scalar Potentials

The techniques used in section 3.2, to obtain expressions for transverse trace-free tensors, in terms of scalar potentials in flat space, are applied to a more general axially-symmetric 3-space metric in section 3.3.1. One scalar potential is then derived in section 3.3.2, though the extra complications of having a non-flat metric, restrict the possibility of a second potential.

The remaining equations are reduced in section 3.3.3, to give a second order partial differential equation in two of the tensor components. The metric is simplified in section 3.3.4, but the simplification only proves to shift the *coordinates* of the flat space metric, and not change the shape of the space itself.

3.3.1 Transverse Trace-Free Tensor with Brill Metric

A general axially symmetric 3-space metric can be given by the Brill wave metric from *Brill* [13] (equation (3)), which was credited there to H. Bondi. In cylindrical-polar type coordinates $(\rho,\ z,\ \phi)$, the conformal part of this metric can be expressed as:

$$\gamma_{ab} \;=\; \begin{pmatrix} e^{2Aq(\rho,z)} & 0 & 0 \\ 0 & e^{2Aq(\rho,z)} & 0 \\ 0 & 0 & \rho^2 \end{pmatrix} , \tag{3.3.1}$$

for an arbitrary scalar A and differential function $q(\rho, z)$ such that:

$$q\mid_{\rho=0} \;=\; \partial_\rho\, q\mid_{\rho=0} \;=\; 0\ , \tag{3.3.2}$$

and that it decays faster than $\frac{1}{r}$ at infinity, and is reasonably differential.

The metric given by (3.3.1) is conformally related to the metric of *Baker & Puzio* [5], see equation (3.1.1), with the component variables related by:

$$\begin{aligned} A \;&=\psi^4\, e^{2Aq}, \qquad & B \;&=\; \psi^4\, \rho^2\ , \\ \Leftrightarrow \qquad \psi^4 \;&=\frac{B}{\rho^2}, \qquad & e^{2Aq} \;&=\; \frac{A}{B}\,\rho^2\ , \end{aligned} \tag{3.3.3}$$

where the conformal factor must be a positive differential function, the exponential is by construction positive, and A and B are defined by *Baker & Puzio* to be positive everywhere on the 3-space. The coordinate systems (ρ, z) and (x, y) are also deemed to be equivalent, since in flat space, (ρ, z) describes a cartesian bases of the half-plane for each “slice” of ϕ, which is how the (x, y) coordinates are defined in *Baker & Puzio*.

Since the metric is diagonal, the connection coefficients are again given by (3.2.2), with non-zero terms:

$$\begin{aligned}
\Gamma^{\rho}_{\rho\rho} &= A\,\partial_\rho\,q\,, & \Gamma^{z}_{\rho\rho} &= -A\,\partial_z\,q\,, & & \\
\Gamma^{\rho}_{\rho z} &= \Gamma^{\rho}_{z\rho} = A\,\partial_z\,q\,, & \Gamma^{z}_{\rho z} &= \Gamma^{z}_{z\rho} = A\,\partial_\rho\,q\,, & \Gamma^{\phi}_{\rho\phi} &= \Gamma^{\phi}_{\phi\rho} = \frac{1}{\rho}\,,\\
\Gamma^{\rho}_{zz} &= -A\,\partial_\rho\,q\,, & \Gamma^{z}_{zz} &= A\,\partial_z\,q\,, & & \\
\Gamma^{\rho}_{\phi\phi} &= -\rho\,e^{-2Aq}\,. & & & &
\end{aligned} \tag{3.3.4}$$

The divergence of a tensor T^{ab} with respect to the Brill metric (3.3.1):

$$D_b\,T^{ab} = \partial_b\,T^{ab} + \Gamma^{a}_{bc}\,T^{cb} + \Gamma^{b}_{bc}\,T^{ac}\,, \tag{3.3.5}$$

can be given explicitly, using the connection coefficients (3.3.4) above. Separating the equations for each value of a:

$$\begin{aligned}
D_bT^{\rho b} &= \partial_bT^{\rho b} + \Gamma^{\rho}_{bc}\,T^{cb} + \Gamma^{b}_{bc}\,T^{\rho c}\\
&= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} + \partial_\phi T^{\rho\phi}\\
&\quad + (2\,\Gamma^{\rho}_{\rho\rho} + \Gamma^{z}_{\rho z} + \Gamma^{\phi}_{\rho\phi})\,T^{\rho\rho} + (3\,\Gamma^{\rho}_{\rho z} + \Gamma^{z}_{zz})\,T^{\rho z} + \Gamma^{\rho}_{zz}\,T^{zz} + \Gamma^{\rho}_{\phi\phi}\,T^{\phi\phi}\\
&= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} + \partial_\phi T^{\rho\phi}\\
&\quad + (3A\,\partial_\rho q + \frac{1}{\rho})\,T^{\rho\rho} + 4A\,\partial_z q\,T^{\rho z} - A\,\partial_\rho q\,T^{zz} - \rho\,e^{-2Aq}\,T^{\phi\phi}\,,
\end{aligned}$$

$$\begin{aligned}
D_bT^{zb} &= \partial_bT^{zb} + \Gamma^{z}_{bc}\,T^{cb} + \Gamma^{b}_{bc}\,T^{zc}\\
&= \partial_\rho T^{z\rho} + \partial_z T^{zz} + \partial_\phi T^{z\phi}\\
&\quad + \Gamma^{z}_{\rho\rho}\,T^{\rho\rho} + (\Gamma^{\rho}_{\rho z} + 2\,\Gamma^{z}_{zz})\,T^{zz} + (3\,\Gamma^{z}_{\rho z} + \Gamma^{\rho}_{\rho\rho} + \Gamma^{\phi}_{\rho\phi})\,T^{\rho z}\\
&= \partial_\rho T^{z\rho} + \partial_z T^{zz} + \partial_\phi T^{z\phi} - A\,\partial_z q\,T^{\rho\rho} + 3A\,\partial_z q\,T^{zz} + (4A\,\partial_\rho q + \frac{1}{\rho})\,T^{\rho z}\,,
\end{aligned}$$

$$\begin{aligned}
D_bT^{\phi b} &= \partial_bT^{\phi b} + \Gamma^{\phi}_{bc}\,T^{cb} + \Gamma^{b}_{bc}\,T^{\phi c}\\
&= \partial_\rho T^{\phi\rho} + \partial_z T^{\phi z} + \partial_\phi T^{\phi\phi} + (\Gamma^{\rho}_{\rho\rho} + \Gamma^{z}_{\rho z} + 3\,\Gamma^{\phi}_{\rho\phi})\,T^{\rho\phi} + (\Gamma^{\rho}_{\rho z} + \Gamma^{z}_{zz})\,T^{z\phi}\\
&= \partial_\rho T^{\phi\rho} + \partial_z T^{\phi z} + \partial_\phi T^{\phi\phi} + (2A\,\partial_\rho q + \frac{3}{\rho})\,T^{\rho\phi} + 2A\,\partial_z q\,T^{z\phi}\,.
\end{aligned} \tag{3.3.6}$$

Assuming T^{ab} to be transverse, trace-free, and axially symmetric, see (3.2.8):

$$D_bT^{ab} = 0\,, \tag{3.3.7a}$$

$$\gamma_{ab}T^{ab} = 0\,, \tag{3.3.7b}$$

$$\partial_\phi T^{ab} = 0\,. \tag{3.3.7c}$$

Hence, from the divergence of T^{ab} (3.3.6), the metric (3.3.1), and equation (3.3.7c), the transverse and trace-free equations (3.3.7a) and (3.3.7b) are given by:

$$\begin{aligned} 0 &= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} + (3A\ \partial_\rho q + \frac{1}{\rho})\ T^{\rho\rho} + 4A\ \partial_z q\ T^{\rho z} \\ &\quad - A\ \partial_\rho q\ T^{zz} - \rho\ e^{-2Aq}\ T^{\phi\phi}\ , \end{aligned} \tag{3.3.8a}$$

$$0 = \partial_\rho T^{z\rho} + \partial_z T^{zz} - A\ \partial_z q\ T^{\rho\rho} + 3A\ \partial_z q\ T^{zz} + (4A\ \partial_\rho q + \frac{1}{\rho})\ T^{\rho z}\ , \tag{3.3.8b}$$

$$0 = \partial_\rho T^{\phi\rho} + \partial_z T^{\phi z} + (2A\ \partial_\rho q + \frac{3}{\rho})\ T^{\rho\phi} + 2A\ \partial_z q\ T^{z\phi}\ , \tag{3.3.8c}$$

$$0 = e^{2Aq}\ T^{\rho\rho} + e^{2Aq}\ T^{zz} + \rho^2\ T^{\phi\phi}\ . \tag{3.3.8d}$$

3.3.2 Time-Rotation Potential

Taking equation (3.3.8c):

$$\begin{aligned} 0 &= \partial_\rho T^{\phi\rho} + \partial_z T^{\phi z} + (2A\ \partial_\rho q + \frac{3}{\rho})\ T^{\rho\phi} + 2A\ \partial_z q\ T^{z\phi} \\ &= e^{2Aq}\ \partial_\rho T^{\rho\phi} + T^{\rho\phi}\ \partial_\rho e^{2Aq} + T^{\rho\phi}\ e^{2Aq}\ \frac{1}{\rho^3}\partial_\rho \rho^3 \\ &\quad +\ e^{2Aq}\ \partial_z T^{z\phi} + T^{z\phi}\ \partial_z e^{2Aq} \\ &= \frac{1}{\rho^3}\ \partial_\rho\ (\rho^3\ e^{2Aq}\ T^{\rho\phi}) + \frac{1}{\rho^3}\ \partial_z\ (\rho^3\ e^{2Aq}\ T^{z\phi})\ , \end{aligned}$$

$$\Leftrightarrow \qquad \partial_\rho\ (\rho^3\ e^{2Aq}\ T^{\rho\phi})\ =\ \partial_z\ (-\rho^3\ e^{2Aq}\ T^{z\phi})\ , \tag{3.3.9}$$

hence, by the equivalence of mixed partial derivatives, there must exist a scalar potential w such that:

$$\begin{aligned} \partial_z\ w &= \rho^3\ e^{2Aq}\ T^{\rho\phi}, \\ \partial_\rho\ w &= -\ \rho^3\ e^{2Aq}\ T^{z\phi}, \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} T^{\rho\phi} &= \rho^{-3}\ e^{-2Aq}\ \partial_z w, \\ T^{z\phi} &= -\ \rho^{-3}\ e^{-2Aq}\ \partial_\rho w. \end{aligned} \tag{3.3.10}$$

With the space-time assumed to have a "time-rotation" symmetry, *Brandt & Seidel* [12] show these components to give the only non-zero components of the extrinsic curvature. In this case, the tensor agrees exactly with the curvature given by *Dain* [19] in equation (3.1.15).

The components given by (3.3.10) can also be seen to agree with the curvature components of *Baker & Puzio* [5], given by equation (3.1.6), by simply substituting from the relations (3.3.3).

3.3.3 Equations for Tensors Free of Time-Rotation Symmetry

Taking equation (3.3.8b), and manipulating in a similar manner to (3.2.22), for the flat cylindrical space:

$$
\begin{aligned}
0 &= \partial_\rho T^{z\rho} + \partial_z T^{zz} - A\ \partial_z q\ T^{\rho\rho} + 3A\ \partial_z q\ T^{zz} + (4A\ \partial_\rho q + \frac{1}{\rho})\ T^{\rho z} \\
&= e^{2Aq}\ \partial_\rho T^{\rho z} + 2\ \partial_\rho e^{2Aq}\ T^{\rho z} + \frac{1}{\rho}\ e^{2Aq}\ T^{\rho z} \\
&\quad + e^{2Aq}\ \partial_z T^{zz} + \frac{3}{2}\ \partial_z e^{2Aq}\ T^{zz} - \frac{1}{2}\ \partial_z e^{2Aq}\ T^{\rho\rho} \\
&= \frac{1}{\rho^2}\ \partial_\rho\ (\rho^2\ e^{2Aq}\ T^{\rho z}) + \frac{1}{\rho^2}\ \partial_z\ (\rho^2\ e^{2Aq}\ T^{zz}) \\
&\quad + T^{\rho z}\partial_\rho e^{2Aq} - \frac{1}{2\rho^2}\ e^{2Aq}\ T^{\rho z}\ \partial_\rho \rho^2 + \frac{1}{2}\ T^{zz}\ \partial_z e^{2Aq} - \frac{1}{2}\ T^{\rho\rho}\ \partial_z e^{2Aq}\ .
\end{aligned} \tag{3.3.11}
$$

However, it can be seen from the second line of the final equation, that the terms $T^{\rho z}$ and T^{zz} cannot be related so easily with a non-flat metric.

Now taking equation (3.3.8a), and using (3.3.8d) to remove $T^{\phi\phi}$ from the equation:

$$
\begin{aligned}
0 &= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} + (3A\ \partial_\rho q + \frac{1}{\rho})\ T^{\rho\rho} + 4A\ \partial_z q\ T^{\rho z} \\
&\quad - A\ \partial_\rho q\ T^{zz} - \rho\ e^{-2Aq}\ T^{\phi\phi} \\
&= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} + (3A\ \partial_\rho q + \frac{1}{\rho})\ T^{\rho\rho} + 4A\ \partial_z q\ T^{\rho z} - A\ \partial_\rho q\ T^{zz} \\
&\quad + \frac{1}{\rho}\ T^{zz} + \frac{1}{\rho}\ T^{\rho\rho} \\
&= e^{2Aq}\ \partial_\rho T^{\rho\rho} + \frac{3}{2}\ T^{\rho\rho}\ \partial_\rho e^{2Aq} + e^{2Aq}\ T^{\rho\rho}\ \frac{2}{\rho} \\
&\quad + e^{2Aq}\ \partial_z T^{\rho z} + 2\ T^{\rho z}\ \partial_z e^{2Aq} \\
&\quad - \frac{1}{2}\ T^{zz}\ \partial_\rho e^{2Aq} + \frac{1}{\rho}\ e^{2Aq}\ T^{zz} \\
&= \frac{1}{\rho^2}\ \partial_\rho\ (\rho^2\ e^{2Aq}\ T^{\rho\rho}) + \frac{1}{\rho^2}\ \partial_z\ (\rho^2\ e^{2Aq}\ T^{\rho z}) \\
&\quad + \frac{1}{2}\ T^{\rho\rho}\ \partial_\rho e^{2Aq} + T^{\rho z}\ \partial_z e^{2Aq} - \frac{1}{2}\ \rho^2\ T^{zz}\ \partial_\rho\ (\rho^{-2} e^{2Aq})\ ,
\end{aligned} \tag{3.3.12}
$$

with the equation manipulated similar to (3.2.26). The non-flat connection coefficients again make the equation more complicated, than in the flat 3-space in cylindrical coordinates.

Since both equations (3.3.11) and (3.3.12) contain similar terms for the component $T^{\rho z}$, if both equations can be manipulated to give equivalent forms of the $T^{\rho z}$ terms, than the remaining $T^{\rho\rho}$ and T^{zz} parts of each equation can be equated. This will at least remove the dependance of one of the remaining components.

Beginning with equation (3.3.11), and bringing all of the $T^{\rho z}$ terms to one side:

$$\begin{aligned} 0 \;=\; & \frac{1}{\rho^2}\,\partial_\rho\left(\rho^2\; e^{2Aq}\; T^{\rho z}\right) + \frac{1}{\rho^2}\,\partial_z\left(\rho^2\; e^{2Aq}\; T^{zz}\right) \\ & + T^{\rho z}\;\partial_\rho e^{2Aq} - \frac{1}{2\rho^2}\; e^{2Aq}\; T^{\rho z}\;\partial_\rho \rho^2 + \frac{1}{2}\; T^{zz}\;\partial_z e^{2Aq} - \frac{1}{2}\; T^{\rho\rho}\;\partial_z e^{2Aq}\;, \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \qquad & \partial_\rho\left(\rho^2\; e^{2Aq}\; T^{\rho z}\right) + \rho^2\; T^{\rho z}\;\partial_\rho e^{2Aq} - \frac{1}{2}\; e^{2Aq}\; T^{\rho z}\;\partial_\rho \rho^2 \\ & = -\,\partial_z\left(\rho^2\; e^{2Aq}\; T^{zz}\right) - \frac{1}{2}\,\rho^2\; T^{zz}\;\partial_z e^{2Aq} + \frac{1}{2}\,\rho^2\; T^{\rho\rho}\;\partial_z e^{2Aq}\;. \end{aligned} \tag{3.3.13}$$

Since the $T^{\rho z}$ terms have derivatives with respect to ρ here, and derivatives with respect to z in equation (3.3.12), both sides of equation (3.3.13) above, are integrated with respect to ρ:

$$\begin{aligned} & \int \partial_\rho\left(\rho^2\; e^{2Aq}\; T^{\rho z}\right)\, d\rho + \int \rho^2\; T^{\rho z}\;\partial_\rho e^{2Aq}\; d\rho - \frac{1}{2}\int e^{2Aq}\; T^{\rho z}\;\partial_\rho \rho^2\; d\rho \\ & \quad = -\int \partial_z\left(\rho^2\; e^{2Aq}\; T^{zz}\right)\, d\rho - \frac{1}{2}\int \rho^2\; T^{zz}\;\partial_z e^{2Aq}\; d\rho \\ & \qquad + \frac{1}{2}\int \rho^2\; T^{\rho\rho}\;\partial_z e^{2Aq}\; d\rho\;, \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \qquad & \rho^2\; e^{2Aq}\; T^{\rho z} + \int \left(\rho^2\; T^{\rho z}\right)\; de^{2Aq} - \frac{1}{2}\int \left(e^{2Aq}\; T^{\rho z}\right)\; d\rho^2 \\ & = -\int \partial_z\left(\rho^2\; e^{2Aq}\; T^{zz}\right)\, d\rho - \frac{1}{2}\int \left(\rho^2\; T^{zz}\;\partial_z e^{2Aq}\right)\; d\rho \\ & \quad + \frac{1}{2}\int \left(\rho^2\; T^{\rho\rho}\;\partial_z e^{2Aq}\right)\; d\rho\;, \end{aligned} \tag{3.3.14}$$

where the chain rule has been used to remove the dependence on ρ for the integrals on the left hand side.

Taking now equation (3.3.12), and again bringing the $T^{\rho z}$ terms to one side:

$$\begin{aligned}
0 \ =& \ \frac{1}{\rho^2}\,\partial_\rho\left(\rho^2\ e^{2Aq}\ T^{\rho\rho}\right) + \frac{1}{\rho^2}\,\partial_z\left(\rho^2\ e^{2Aq}\ T^{\rho z}\right) \\
&+ \frac{1}{2}\,T^{\rho\rho}\ \partial_\rho e^{2Aq} + T^{\rho z}\ \partial_z e^{2Aq} - \frac{1}{2}\,\rho^2\ T^{zz}\ \partial_\rho\left(\rho^{-2}e^{2Aq}\right)\ ,
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \qquad & \frac{1}{\rho^2}\,\partial_z\left(\rho^2\ e^{2Aq}\ T^{\rho z}\right) + T^{\rho z}\ \partial_z e^{2Aq} \\
&= \ -\frac{1}{\rho^2}\,\partial_\rho\left(\rho^2\ e^{2Aq}\ T^{\rho\rho}\right) - \frac{1}{2}\,T^{\rho\rho}\ \partial_\rho e^{2Aq} + \frac{1}{2}\,\rho^2\ T^{zz}\ \partial_\rho\left(\rho^{-2}e^{2Aq}\right)\ .
\end{aligned} \tag{3.3.15}$$

With both sides of equation (3.3.15) integrated with respect to z, there is still a difference with equation (3.3.14). However, the missing term can be given by first adding a term involving the derivative $\partial_z\rho^2$, which itself evaluates to zero. Hence equation (3.3.15) is equivalent to:

$$\begin{aligned}
&\partial_z\left(\rho^2\ e^{2Aq}\ T^{\rho z}\right) + \rho^2\ T^{\rho z}\ \partial_z e^{2Aq} - \frac{1}{2}\ e^{2Aq}\ T^{\rho z}\ \partial_z\rho^2 \\
&= \ -\partial_\rho\left(\rho^2\ e^{2Aq}\ T^{\rho\rho}\right) - \frac{1}{2}\,\rho^2\ T^{\rho\rho}\ \partial_\rho e^{2Aq} + \frac{1}{2}\,\rho^4\ T^{zz}\ \partial_\rho\left(\rho^{-2}e^{2Aq}\right)\ ,
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \qquad & \int\partial_z\left(\rho^2\ e^{2Aq}\ T^{\rho z}\right)\,dz + \int\rho^2\ T^{\rho z}\ \partial_z e^{2Aq}\ dz - \frac{1}{2}\int e^{2Aq}\ T^{\rho z}\ \partial_z\rho^2\ dz \\
&= \ -\int\partial_\rho\left(\rho^2\ e^{2Aq}\ T^{\rho\rho}\right)\,dz - \frac{1}{2}\int\rho^2\ T^{\rho\rho}\ \partial_\rho e^{2Aq}\ dz \\
&\quad + \frac{1}{2}\int\rho^4\ T^{zz}\ \partial_\rho\left(\rho^{-2}e^{2Aq}\right)\,dz\ ,
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \qquad & \rho^2\ e^{2Aq}\ T^{\rho z} + \int\left(\rho^2\ T^{\rho z}\right)\,de^{2Aq} - \frac{1}{2}\int\left(e^{2Aq}\ T^{\rho z}\right)\,d\rho^2 \\
&= \ -\int\partial_\rho\left(\rho^2\ e^{2Aq}\ T^{\rho\rho}\right)\,dz - \frac{1}{2}\int\left(\rho^2\ T^{\rho\rho}\ \partial_\rho e^{2Aq}\right)\,dz \\
&\quad + \frac{1}{2}\int\left(\rho^4\ T^{zz}\ \partial_\rho\left(\rho^{-2}e^{2Aq}\right)\right)\,dz\ ,
\end{aligned} \tag{3.3.16}$$

again, using the chain rule to remove the z dependance from the integrals on the left hand side.

Since the left hand sides of both (3.3.14) and (3.3.16) are equivalent, the right hand sides can be equated, to get a single equation depending on the components $T^{\rho\rho}$ and T^{zz} alone:

$$\begin{aligned}
& -\int \partial_z \left(\rho^2\, e^{2Aq}\, T^{zz}\right) d\rho - \frac{1}{2}\int \left(\rho^2\, T^{zz}\, \partial_z\, e^{2Aq}\right)\, d\rho \\
& +\frac{1}{2}\int \left(\rho^2\, T^{\rho\rho}\, \partial_z\, e^{2Aq}\right)\, d\rho \\
= & -\int \partial_\rho \left(\rho^2\, e^{2Aq}\, T^{\rho\rho}\right) dz - \frac{1}{2}\int \left(\rho^2\, T^{\rho\rho}\, \partial_\rho\, e^{2Aq}\right)\, dz \\
& +\frac{1}{2}\int \left(\rho^4\, T^{zz}\, \partial_\rho \left(\rho^{-2} e^{2Aq}\right)\right)\, dz \ .
\end{aligned} \tag{3.3.17}$$

Differentiating both sides with respect to both ρ and z, gives a second order partial differential equation in $T^{\rho\rho}$ and T^{zz}:

$$\begin{aligned}
& \partial_z\partial_z \left(\rho^2\, e^{2Aq}\, T^{zz}\right) + \frac{1}{2}\partial_z \left(\rho^2\, T^{zz}\, \partial_z\, e^{2Aq}\right) - \frac{1}{2}\partial_z \left(\rho^2\, T^{\rho\rho}\, \partial_z\, e^{2Aq}\right) \\
= & \ \partial_\rho\partial_\rho \left(\rho^2\, e^{2Aq}\, T^{\rho\rho}\right) + \frac{1}{2}\partial_\rho \left(\rho^2\, T^{\rho\rho}\, \partial_\rho\, e^{2Aq}\right) - \frac{1}{2}\partial_\rho \left(\rho^4\, T^{zz}\, \partial_\rho \left(\rho^{-2} e^{2Aq}\right)\right) \ .
\end{aligned} \tag{3.3.18}$$

The third term on the left hand side can be adjusted, since ρ is seen as a constant for derivatives with respect to z:

$$\begin{aligned}
& \partial_z\partial_z \left(\rho^2\, e^{2Aq}\, T^{zz}\right) + \frac{1}{2}\partial_z \left(\rho^2\, T^{zz}\, \partial_z\, e^{2Aq}\right) - \frac{1}{2}\partial_z \left(\rho^4\, T^{\rho\rho}\, \partial_z \left(\rho^{-2} e^{2Aq}\right)\right) \\
= & \ \partial_\rho\partial_\rho \left(\rho^2\, e^{2Aq}\, T^{\rho\rho}\right) + \frac{1}{2}\partial_\rho \left(\rho^2\, T^{\rho\rho}\, \partial_\rho\, e^{2Aq}\right) - \frac{1}{2}\partial_\rho \left(\rho^4\, T^{zz}\, \partial_\rho \left(\rho^{-2} e^{2Aq}\right)\right) \ ,
\end{aligned} \tag{3.3.19}$$

giving a symmetry to both sides of the equation.

Equation (3.3.19) can also be represented specifically in terms of the components $T^{\rho\rho}$, T^{zz} and their derivatives:

$$\begin{aligned}
& \partial_{\rho\rho}T^{\rho\rho}\ \left(\rho^2 e^{2Aq}\right) && = \quad \partial_{zz}T^{zz}\ \left(\rho^2 e^{2Aq}\right) \\
& + \partial_\rho T^{\rho\rho}\ \left(4\ \rho\ e^{2Aq} + \frac{5}{2}\rho^2\partial_\rho e^{2Aq}\right) && \quad + \partial_z T^{zz}\ \left(\frac{5}{2}\rho^2\partial_z e^{2Aq}\right) \\
& + \partial_z T^{\rho\rho}\ \left(\frac{1}{2}\rho^2\partial_z e^{2Aq}\right) && \quad + \partial_\rho T^{zz}\ \left(\frac{1}{2}\rho^2\partial_\rho e^{2Aq} - \rho\ e^{2Aq}\right) \\
& \quad + T^{\rho\rho}\ \Big(\frac{3}{2}\rho^2\partial_{\rho\rho}e^{2Aq} + 5\ \rho\ \partial_\rho e^{2Aq} && \quad\quad + T^{zz}\ \Big(\frac{3}{2}\rho^2\partial_{zz}e^{2Aq} \\
& \qquad + \frac{1}{2}\rho^2\partial_{zz}e^{2Aq} + 2\ e^{2Aq}\Big) && \qquad + \frac{1}{2}\rho^2\partial_{\rho\rho}e^{2Aq} - e^{2Aq}\Big) \ .
\end{aligned} \tag{3.3.20}$$

Unfortunately, unlike the flat space tensors (3.2.15) and (3.2.27), finding a relation between the components $T^{\rho\rho}$ and T^{zz} in a space with metric (3.3.1), involves solving one of equations (3.3.18) to (3.3.20). These equations are each second order partial differential equations, and hence require two separate boundary conditions to give a unique solution. These boundary conditions require more information about the system, for example, asymptotic flatness, or behavior close to the axis of symmetry.

3.3.4 Setting $q(\rho, z)$ to One

Setting $q(\rho, z)$ to one, gives $e^{2Aq} = e^{2A}$, which becomes zero when differentiated. Equation (3.3.8b), then reduces to:

$$\begin{aligned} 0 &= \partial_\rho T^{z\rho} + \partial_z T^{zz} - A\ \partial_z q\ T^{\rho\rho} + 3A\ \partial_z q\ T^{zz} + (4A\ \partial_\rho q + \frac{1}{\rho})\ T^{\rho z} \\ &= \partial_\rho T^{z\rho} + \partial_z T^{zz} + \frac{1}{\rho}\ T^{\rho z} \\ &= \frac{1}{\rho}\ \partial_z(\rho\ T^{zz}) + \frac{1}{\rho}\ \partial_\rho(\rho\ T^{z\rho})\ , \end{aligned} \tag{3.3.21}$$

$$\Leftrightarrow \qquad \partial_z(-\rho\ T^{zz}) \ = \ \partial_\rho(\rho\ T^{z\rho})\ ,$$

and by the equivalence of mixed partial derivatives, there must exist a scalar potential v such that:

$$\begin{aligned} \partial_\rho v &= -\rho\ T^{zz}\ , \\ \partial_z v &= \rho\ T^{z\rho}\ , \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} T^{zz} &= -\rho^{-1}\ \partial_\rho v\ , \\ T^{z\rho} &= \rho^{-1}\ \partial_z v\ . \end{aligned} \tag{3.3.22}$$

Equation (3.3.8a), using also (3.3.8d) and (3.3.22), reduces to:

$$\begin{aligned} 0 &= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} + (3A\ \partial_\rho q + \frac{1}{\rho})\ T^{\rho\rho} + 4A\ \partial_z q\ T^{\rho z} \\ &\quad - A\ \partial_\rho q\ T^{zz} - \rho\ e^{-2Aq}\ T^{\phi\phi} \\ &= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} + \frac{1}{\rho}\ T^{\rho\rho} - \rho\ e^{-2Aq}\ T^{\phi\phi} \\ &= \partial_\rho T^{\rho\rho} + \partial_z T^{\rho z} + \frac{2}{\rho}\ T^{\rho\rho} + \frac{1}{\rho}\ T^{zz} \\ &= \frac{1}{\rho^2}(\rho^2\ T^{\rho\rho}) + \frac{1}{\rho}\ \partial_{zz} v - \frac{1}{\rho^2}\ \partial_\rho v\ , \end{aligned}$$

$$\Leftrightarrow \qquad \partial_\rho \left(\rho^2\ T^{\rho\rho}\right) \ = \ \partial_\rho v - \rho\ \partial_{zz} v\ ,$$

$$\Leftrightarrow \qquad T^{\rho\rho} \ = \ \frac{1}{\rho^2} \int \left[\partial_\rho v - \rho\ \partial_{zz} v\right] d\rho\ . \tag{3.3.23}$$

This is exactly the same as the flat space tensor, in cylindrical coordinates (3.2.27), for the 2-surface described by the coordinate vector fields ρ and z. The factor e^{2A} acts as a *re-scaling* of the ρ and z coordinates, with respect to the azimuthal ϕ coordinate. Hence the term $\rho^{-3}\ e^{-2A}$ still arises in the ϕ terms of the tensor, given by equation (3.3.10), but the metric still describes a flat space.

3.4 Relating Flat Space Potentials to Bowen-York Curvature

Since the Bowen-York initial conditions, described in section 2.2.5, are axially-symmetric, and are given by a conformally flat spatial metric, the conformal curvature tensor should be given by an appropriate choice of potentials for the tensors derived in section 3.2.

3.4.1 Bowen-York Curvature with Angular Momentum Only

Taking the case of zero linear momentum, the conformal Bowen-York curvature is given, from equation (2.2.61), by:

$$\bar{K}_{ab} \;=\; \frac{3}{r^3}\,\left(\epsilon_{acd}\, q_b + \epsilon_{bcd}\, q_a\right) q^c J^d \;, \tag{3.4.1}$$

depending on the angular momentum J^a alone, with q^a the unit normal to a sphere of constant radius, and ϵ_{abc} the Levi-Civita alternating tensor described by equation (2.2.57).

In cylindrical type coordinates $(\rho,\ z,\ \phi)$, with the angular momentum directed in the axial direction, the unit space-like normal q^a and angular momentum vector are given by:

$$q^a \;=\; q_a \;=\; \frac{(\rho,\ z,\ 0)}{\sqrt{\rho^2+z^2}}\;, \qquad J^a \;=\; (0,\ J,\ 0)\;. \tag{3.4.2}$$

Recalling from page 87, that the Levi-Civita tensor has a reversed orientation for the coordinates given in this order, the non-zero terms of the conformal Bowen-York curvature are given by:

$$\bar{K}_{\rho\phi} \;=\; \bar{K}_{\phi\rho} \;=\; \frac{3J\rho^3}{r^5}\;, \tag{3.4.3a}$$

$$\bar{K}_{z\phi} \;=\; \bar{K}_{\phi z} \;=\; \frac{3J\rho^2 z}{r^5}\;, \tag{3.4.3b}$$

and raising the indices, for comparison with (3.2.27), the non-zero components are given by:

$$\bar{K}^{\rho\phi} \;=\; \bar{K}^{\phi\rho} \;=\; \frac{3J\rho}{r^5}\;, \tag{3.4.4a}$$

$$\bar{K}^{z\phi} \;=\; \bar{K}^{\phi z} \;=\; \frac{3Jz}{r^5}\;. \tag{3.4.4b}$$

Comparing to Scalar Potential

Since the Bowen-York conformal curvature tensor is transverse, trace-free and axially symmetric, in cylindrical coordinates (3.4.4), it must be representable by the transverse trace-free tensor (3.2.27), derived in section 3.2.2, with an appropriate choice of potentials.

To find the necessary choice of potentials, the components of the tensor (3.2.27) are set equal to those of the Bowen-York curvature (3.4.4):

$$
\begin{aligned}
& T^{\rho\phi} = \bar{K}^{\rho\phi} , && T^{z\phi} = \bar{K}^{z\phi} , \\
\Leftrightarrow \quad & \frac{1}{\rho^3}\partial_z Y = \frac{3J\rho}{r^5} , && -\frac{1}{\rho^3}\partial_\rho Y = \frac{3Jz}{r^5} , \\
\Leftrightarrow \quad & \partial_z Y = \frac{3J\rho^4}{r^5} , && \partial_\rho Y = -\frac{3J\rho^3 z}{r^5} ,
\end{aligned}
\tag{3.4.5}
$$

with the remaining tensor components equal to zero, implying that the potential X must be set to zero. Integrating each equation in (3.4.5), with respect to z and ρ respectively, leads to the solution:

$$
Y = J\,\frac{3\rho^2 z + 2z^3}{r^3} , \tag{3.4.6}
$$

plus a constant of integration, not dependent on ρ or z, which doesn't effect the tensor due to the derivatives of the potential.

Hence, taking the transverse trace-free tensor (3.2.27), and choosing the two scalar potentials as:

$$
X = 0 , \tag{3.4.7a}
$$

$$
Y = J\,\frac{3\rho^2 z + 2z^3}{r^3} , \tag{3.4.7b}
$$

gives the Bowen-York conformal extrinsic curvature, with angular momentum J, in cylindrical coordinates (3.4.4). Due to X being zero, this can easily be translated into spherical coordinates, with the potential $V = 0$, and W, from relations (3.2.48) and (3.2.28), given by:

$$
\begin{aligned}
W &= -J(3\sin^2\theta\ \cos\theta + 2\cos^3\theta) \\
&= J(\cos^3\theta - 3\cos\theta) ,
\end{aligned}
\tag{3.4.8}
$$

which agrees with a calculation in *Dain et al.* [21] (equation (19)), for the curvature tensor (3.1.15), derived in *Dain* [19].

3.4.2 Bowen-York Curvature with Linear Momentum Only

The conformal Bowen-York extrinsic curvature for a system with no angular momentum is given from (2.2.60) by:

$$\begin{aligned} \bar{K}^{\pm}_{ab} &= \frac{3}{2r^2}\;[P_a q_b + P_b q_a - (\bar{\gamma}_{ab} - q_a q_b)P^c q_c] \\ &\mp \frac{3a^2}{2\;r^4}\,[P_a q_b + P_b q_a + (\bar{\gamma}_{ab} - 5 q_a q_b)P^c q_c]\;\;, \end{aligned} \tag{3.4.9}$$

where P^a denotes the linear momentum of a single source, q^a the unit normal to a sphere of constant radius and a an arbitrary constant.

The linear momentum in the Bowen-York curvature $\bar{K}^{\pm}_{ab}$, can only give an axially-symmetric tensor, if the momentum is directed along the axis. Hence, in cylindrical type coordinates $(\rho,\ z,\ \phi)$, the linear momentum vector and unit space-like normal q^a are given by:

$$P^a \;=\; (0,\ P,\ 0)\ , \qquad q^a \;=\; q_a \;=\; \frac{(\rho,\ z,\ 0)}{\sqrt{\rho^2+z^2}}\ , \tag{3.4.10}$$

giving the components of the conformal Bowen-York curvature:

$$\bar{K}^{\pm}_{\rho\rho} \;=\frac{3Pz}{2\;r^5}(-r^2+\rho^2)\;\mp\;\frac{3a^2Pz}{2\;r^7}(r^2-5\;\rho^2)\ , \tag{3.4.11a}$$

$$\bar{K}^{\pm}_{zz} \;=\frac{3Pz}{2\;r^5}(r^2+z^2)\;\mp\;\frac{3a^2Pz}{2\;r^7}(3\;r^2-5\;z^2)\ , \tag{3.4.11b}$$

$$\bar{K}^{\pm}_{\rho z} \;=\frac{3P\rho}{2\;r^5}(r^2+z^2)\;\mp\;\frac{3a^2P\rho}{2\;r^7}(r^2-5\;z^2)\ , \tag{3.4.11c}$$

$$\bar{K}^{\pm}_{\phi\phi} \;=-\;\frac{3P\rho^2 z}{2\;r^3}\;\mp\;\frac{3a^2P\rho^2 z}{2\;r^5}\ , \tag{3.4.11d}$$

$$\bar{K}^{\pm}_{\rho\phi} \;=0\ , \tag{3.4.11e}$$

$$\bar{K}^{\pm}_{z\phi} \;=0\ , \tag{3.4.11f}$$

with $r\;=\;\sqrt{\rho^2+z^2}$. The zero components here, can be seen by equations (3.4.3), to coincide with the non-zero components for the *angular* Bowen-York curvature.

The indices are again raised, for comparison with (3.2.27), giving the matrix form:

$$\bar{K}^{ab}_{\pm} \;=\; \frac{3\;P}{2\;r^7}\begin{pmatrix} \begin{array}{c}-z^3\;r^2\\ \mp\;a^2 z(z^2-4\rho^2)\end{array} & \begin{array}{c}\rho\;r^2(2\rho^2+z^2)\\ \mp\;a^2\rho(\rho^2-4z^2)\end{array} & 0 \\ \\ \begin{array}{c}\rho\;r^2(2\rho^2+z^2)\\ \mp\;a^2\rho(\rho^2-4z^2)\end{array} & \begin{array}{c}z\;r^2(2\rho^2+z^2)\\ \mp\;a^2 z(3\rho^2-2z^2)\end{array} & 0 \\ \\ 0 & 0 & -\frac{z\;r^4}{\rho^2}\mp a^2\frac{z\;r^2}{\rho^2} \end{pmatrix}\ . \tag{3.4.12}$$

Comparing to Scalar Potential

It can clearly be seen from equation (3.4.12), that the scalar potential Y in (3.2.27), must be chosen to be zero. To find the scalar potential X, which gives the conformal Bowen-York extrinsic curvature (3.4.12), the ρz components for each are first equated:

$$T^{\rho z} = \bar{K}^{\rho z}_{\pm} ,$$

$$\Leftrightarrow \qquad \frac{1}{\rho}\partial_z X = \frac{3P\rho}{2r^5}\left(2\rho^2+z^2\right) \mp \frac{3a^2P\rho}{2r^7}\left(\rho^2-4z^2\right) ,$$

$$\Leftrightarrow \qquad X = \frac{3}{2}P\rho^2\int\frac{1}{r^5}\left(2\rho^2+z^2\right)dz \mp \frac{3}{2}a^2P\rho^2\int\frac{1}{r^7}\left(\rho^2-4z^2\right)dz , \tag{3.4.13}$$

and equating also the zz components:

$$T^{zz} = \bar{K}^{zz}_{\pm} ,$$

$$\Leftrightarrow \qquad -\frac{1}{\rho}\partial_\rho X = \frac{3Pz}{2r^5}\left(2\rho^2+z^2\right) \mp \frac{3a^2Pz}{2r^7}\left(3\rho^2-2z^2\right) ,$$

$$\Leftrightarrow \qquad X = -\frac{3}{2}Pz\int\frac{\rho}{r^5}\left(2\rho^2+z^2\right)d\rho \pm \frac{3}{2}a^2Pz\int\frac{\rho}{r^7}\left(3\rho^2-2z^2\right)d\rho . \tag{3.4.14}$$

Carrying out the two sets of integrals, noting that $r=\sqrt{\rho^2+z^2}$, gives the result:

$$X = P\,\frac{3\rho^2z+4z^3}{2r^3} \mp P\,\frac{3\,\rho^2z}{2r^5} , \tag{3.4.15}$$

which, similar to equation (3.4.6), also contains a constant of integration, not dependent on ρ or z, but which again doesn't effect the tensor due to the derivatives of the potential.

Hence, taking the transverse trace-free tensor (3.2.27), and choosing the two scalar potentials as:

$$X = P\,\frac{3\rho^2z+4z^3}{2r^3} \mp P\,\frac{3\,\rho^2z}{2r^5} , \tag{3.4.16a}$$

$$Y = 0 , \tag{3.4.16b}$$

gives the Bowen-York conformal extrinsic curvature, with *linear* momentum P, in cylindrical coordinates (3.4.12).

3.4.3 Combined Angular and Linear Momentum

The two solutions for the Bowen-York extrinsic curvature can be combined, with both momenta directed along the axis of symmetry, i.e. the z direction, giving the conformal curvature in cylindrical coordinates (ρ, z, ϕ):

$$\bar{K}^{ab}_{\pm} = \frac{3\ P}{2\ r^7} \begin{pmatrix} \begin{matrix} -z^3\ r^2 \\ \mp\ a^2 z(z^2 - 4\rho^2) \end{matrix} & \begin{matrix} \rho\ r^2(2\rho^2 + z^2) \\ \mp\ a^2 \rho(\rho^2 - 4z^2) \end{matrix} & \frac{3J\rho}{r^5} \\ \begin{matrix} \rho\ r^2(2\rho^2 + z^2) \\ \mp\ a^2 \rho(\rho^2 - 4z^2) \end{matrix} & \begin{matrix} z\ r^2(2\rho^2 + z^2) \\ \mp\ a^2 z(3\rho^2 - 2z^2) \end{matrix} & \frac{3Jz}{r^5} \\ \frac{3J\rho}{r^5} & \frac{3Jz}{r^5} & -\frac{z\ r^4}{\rho^2} \mp a^2 \frac{z\ r^2}{\rho^2} \end{pmatrix}, \quad (3.4.17)$$

which is given by the choice of scalar potentials:

$$X = P\ \frac{3\rho^2 z + 4z^3}{2r^3} \mp P\ \frac{3\ \rho^2 z}{2r^5}\ , \quad (3.4.18a)$$

$$Y = J\ \frac{3\rho^2 z + 2z^3}{r^3}\ , \quad (3.4.18b)$$

for the transverse trace-free and axially symmetric tensor (3.2.27), in cylindrical coordinates.

Due to the relation to the Bowen-York curvature, the potential X can be considered to be related to the choice of a general linear momentum, and the potential Y to a general angular momentum for the tensor (3.2.27).

3.5 Regularity of Tensors at the Origin

Both the Bowen-York conformal curvature (3.4.4) and Kerr extrinsic curvature (A.1.36), can be seen to have an inverse "radial" dependence. As a result of this dependence, the tensors become non-regular as the origin is approached, with the components tending to infinity.

The expressions given for transverse trace-free and axially symmetric tensors, in flat space (3.2.15) and (3.2.27), can also be seen to have an inverse "radial" dependence. To find a tensor which is regular at the origin, the scalar potentials need to be chosen to remove the effect of these inverse factors.

The transverse trace-free tensor T^{ab}, is taken in cylindrical coordinates by (3.2.27):

$$T^{ab} = \begin{pmatrix} \frac{1}{\rho^2}\int\left[-\rho\,\partial_{zz}X+\partial_\rho X\right]d\rho & \frac{1}{\rho}\,\partial_z X & \frac{1}{\rho^3}\partial_z Y \\ \frac{1}{\rho}\,\partial_z X & -\frac{1}{\rho}\,\partial_\rho X & -\frac{1}{\rho^3}\partial_\rho Y \\ \frac{1}{\rho^3}\partial_z Y & -\frac{1}{\rho^3}\partial_\rho Y & \frac{1}{\rho^3}\,\partial_\rho X \\ & & -\frac{1}{\rho^4}\int\left[-\rho\,\partial_{zz}X+\partial_\rho X\right]d\rho \end{pmatrix}, \tag{3.5.1}$$

due to its simpler form, compared with the tensor in spherical coordinates (3.2.15).

3.5.1 Choice of Scalar Potential Y

It can be seen by inspection of the $T^{\rho\phi}$ component of (3.5.1), that if the scalar potential Y has a z dependance, than regularity of $T^{\rho\phi}$ at the origin implies that Y must also have a factor ρ^n with $n \geq 3$:

$$Y = \rho^3\,g(z)\,, \qquad \Rightarrow \qquad \begin{aligned} T^{\rho\phi} &= \frac{1}{\rho^3}\,\partial_z\left(\rho^3\,g(z)\right) \\ &= \partial_z\,g(z)\,, \end{aligned} \tag{3.5.2}$$

for a differential function $g(z)$, which is itself regular at the origin. However, with this choice of the potential Y, the component $T^{z\phi}$ is given by:

$$Y = \rho^3\,g(z)\,, \qquad \Rightarrow \qquad \begin{aligned} T^{z\phi} &= -\frac{1}{\rho^3}\,\partial_\rho\left(\rho^3\,g(z)\right) \\ &= -\frac{3}{\rho}\,g(z)\,, \end{aligned} \tag{3.5.3}$$

and therefore the potential Y needs a factor ρ^n with $n \geq 4$, to ensure the component

$T^{z\phi}$ is also regular at the origin:

$$Y \;=\; \rho^4 \; g(z) \;, \qquad \Rightarrow \qquad \begin{aligned} T^{\rho\phi} &= \rho \; \partial_z \; g(z) \;, \\ T^{z\phi} &= -\, 4 \; g(z) \;, \end{aligned} \tag{3.5.4}$$

for some differential function $g(z)$ which is regular at the origin, along with its z derivative. As a result of the ρ^4 factor however, the component $T^{\rho\phi}$ must vanish on the axis of symmetry, and also grows unboundedly large for large values of ρ. The component $T^{z\phi}$ can take *any* value at the origin, depending on the function $g(z)$, though is completely independent of the coordinate ρ.

If the differential function g also depends on the ρ coordinate, though still regular at the origin, the form of the $T^{\rho\phi}$ component remains unchanged, however the derivative in the $T^{z\phi}$ component now acts on the differential function as well:

$$Y \;=\; \rho^4 \; g(\rho, z) \;, \qquad \Rightarrow \qquad \begin{aligned} T^{\rho\phi} &= \rho \; \partial_z \; g(\rho, z) \;, \\ \\ T^{z\phi} &= -\, \frac{1}{\rho^3} \; \partial_\rho \left(\rho^4 \; g(\rho, z) \right) \\ &= -\, 4 \; g(\rho, z) - \rho \; \partial_\rho \; g(\rho, z) \;, \end{aligned} \tag{3.5.5}$$

with the differential function $g(\rho, z)$ now required to be regular at the origin, along with *both* coordinate derivatives. Both the $T^{\rho\phi}$ component and the second part of $T^{z\phi}$ vanish on the axis of symmetry, however an appropriate choice of the function $g(\rho, z)$ can avoid the *unboundedness* of both components as ρ gets large, such as an inverse exponential in ρ.

3.5.2 Choice of Scalar Potential X

There are four components of the tensor (3.5.1) depending on the scalar potential X, with the largest inverse ρ dependence in the $T^{\phi\phi}$ component:

$$\begin{aligned} T^{\phi\phi} &= \frac{1}{\rho^3} \; \partial_\rho X - \frac{1}{\rho^4} \int \left[-\rho \; \partial_{zz} X + \partial_\rho X \right] d\rho \\ &= \frac{1}{\rho^3} \; \partial_\rho X - \frac{1}{\rho^4} X + \frac{1}{\rho^4} \int \left[\rho \; \partial_{zz} X \right] d\rho \;. \end{aligned} \tag{3.5.6}$$

Hence, if X has a dependance on the z coordinate, then it must also have a factor ρ^n with $n \geq 4$, to ensure the first two parts of $T^{\phi\phi}$ in equation (3.5.6) are regular at the origin. This choice of potential, then gives the four X dependant components:

$$X \; = \; \rho^4 \; f(z) \; , \qquad \Rightarrow \qquad \begin{aligned} T^{zz} \; &= \; -4\rho^3 \; f(z) \; , \\ T^{\rho z} \; &= \; \rho^3 \; \partial_z \; f(z) \; , \\ T^{\rho\rho} \; &= \; \rho^2 \; f(z) - \frac{1}{6} \; \rho^4 \; \partial_{zz} \; f(z) \; , \\ T^{\phi\phi} \; &= \; 3 \; f(z) + \frac{1}{6} \; \rho^2 \; \partial_{zz} \; f(z) \; , \end{aligned} \tag{3.5.7}$$

for a differential function $f(z)$ which is regular at the origin, along with its first and second derivatives with respect to z. All of the components are then regular at the origin, and the $T^{\phi\phi}$ component capable of being *non-zero* at the origin. However, as with (3.5.4), the remaining three components, and the second part of $T^{\phi\phi}$ vanish on the axis of symmetry, and grow unboundedly large as ρ increases.

Taking the differential function f to depend on ρ instead of z, the potential X must still take the same form as (3.5.7), giving the components to be:

$$X \; = \; \rho^4 \; f(\rho) \; , \qquad \Rightarrow \qquad \begin{aligned} T^{zz} \; &= \; - \; 4 \; \rho^2 \; f(\rho) - \rho^3 \; \partial_\rho \; f(\rho) \; , \\ T^{\rho z} \; &= \; 0 \; , \\ T^{\rho\rho} \; &= \; \rho^2 \; f(\rho) \; , \\ T^{\phi\phi} \; &= \; 3 \; f(\rho) + \rho \; \partial_\rho \; f(\rho) \; , \end{aligned} \tag{3.5.8}$$

for the differential function $f(\rho)$ regular at the origin, along with its ρ derivative. As with (3.5.7), all of the components are regular at the origin, and $T^{\phi\phi}$ can be non-zero, depending on $f(\rho)$. The remaining components still vanish on the axis of symmetry, but similar to (3.5.5), $f(\rho)$ can be chosen to avoid the unboundedness of the components, for large ρ.

For f to have an open dependence on both ρ and z, difficulties arise due to the integral in the $T^{\rho\rho}$ and $T^{\phi\phi}$ components:

$$X \; = \; \rho^4 \; f(\rho, z) \; , \qquad \Rightarrow \qquad \begin{aligned} T^{\rho\rho} \; &= \; \rho^2 \; f(\rho, z) - \frac{1}{\rho^2} \int \Big[\rho^5 \; \partial_{zz} \; f(\rho, z) \Big] \, d\rho \; , \\ T^{\phi\phi} \; &= \; 3 \; f(\rho, z) + \rho \; \partial_\rho \; f(\rho, z) \\ &\quad + \frac{1}{\rho^4} \int \Big[\rho^5 \; \partial_{zz} \; f(\rho, z) \Big] \, d\rho \; , \end{aligned} \tag{3.5.9}$$

with no guarantee that the evaluated integrals will cancel the inverse ρ factors, and therefore no guarantee of regularity at the origin.

However, if the differential function $f(\rho, z)$ is only *once* differentiable with respect to z, the components can be given by:

$$X = \rho^4 \, f(\rho, z) \; , \qquad \Rightarrow \qquad \begin{aligned} T^{zz} &= -\,4\,\rho^2\, f(\rho,z) - \rho^3\,\partial_\rho\, f(\rho,z) \; , \\ T^{\rho z} &= \rho^3\,\partial_z\, f(\rho,z) \; , \\ T^{\rho\rho} &= \rho^2\, f(\rho,z) \; , \\ T^{\phi\phi} &= 3\, f(\rho,z) + \rho\,\partial_\rho\, f(\rho,z) \; , \end{aligned} \tag{3.5.10}$$

for $f(\rho, z)$ regular at the origin, along with its first derivatives with respect to ρ and z. The components contain mostly the same properties as (3.5.8), though the $T^{\rho z}$ component is no longer zero.

3.5.3 Tensor with Specified Potentials X and Y

The tensor (3.5.1) can now be given, such that it is always regular at the origin, by the matrix representation:

$$T^{ab} = \begin{pmatrix} \rho^2\, f(\rho,z) & \rho^3\,\partial_z\, f(\rho,z) & \rho\,\partial_z\, g(\rho,z) \\ \\ \rho^3\,\partial_z\, f(\rho,z) & \begin{array}{c} -\,4\,\rho^2\, f(\rho,z) \\ -\rho^3\,\partial_\rho\, f(\rho,z) \end{array} & \begin{array}{c} -4\, g(\rho,z) \\ -\rho\,\partial_\rho\, g(\rho,z) \end{array} \\ \\ \rho\,\partial_z\, g(\rho,z) & \begin{array}{c} -4\, g(\rho,z) \\ -\rho\,\partial_\rho\, g(\rho,z) \end{array} & \begin{array}{c} 3\, f(\rho,z) \\ +\rho\,\partial_\rho\, f(\rho,z) \end{array} \end{pmatrix} \; , \tag{3.5.11}$$

with the differential functions $f(\rho, z)$ and $g(\rho, z)$ now playing the parts of the scalar potentials, though with the added restrictions that both functions be regular at the origin, along with their derivatives with respect to both ρ and z, and that the second derivative of $f(\rho, z)$, with respect to z, vanishes.

3.6 Spherically-Symmetric Tensor Products

In a conformally flat space-like hypersurface, the conformal scalar curvature $\bar{R} = 0$, and hence the Hamiltonian constraint (2.2.22) is given by:

$$8\bar{D}^2\psi - \psi\cancelto{0}{\bar{R}} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ab}\bar{A}^{ab} \;=\; -16\pi\psi^5\rho \; . \tag{3.6.1}$$

If a transverse trace-free tensor $\bar{A}_{ab}$ can be found, such that its product $\bar{A}_{ab}\bar{A}^{ab}$ is spherically symmetric, then choosing both the mean curvature K, and the source term ρ to depend only on r, the Hamiltonian constraint above reduces to an *ordinary* differential equation:

$$8\bar{D}^2\psi(r) - \frac{2}{3}\;\psi^5(r)\;K^2 + \psi^{-7}(r)\;\bar{A}_{ab}\bar{A}^{ab} \;=\; -16\pi\psi^5(r)\;\rho \; , \tag{3.6.2}$$

drastically reducing the difficulty in finding the conformal factor.

The scalar product of the tensor T^{ab}, for a diagonal metric, is given by:

$$T^{ab}T_{ab} \;=\; \gamma_{aa}\;\gamma_{bb}\;(T^{ab})^2 \; . \tag{3.6.3}$$

In spherical coordinates, with a flat 3-space metric (3.2.1), any transverse trace-free and axially-symmetric tensor must be given by equation (3.2.15), giving its scalar product (3.6.3) by:

$$\begin{aligned}
T^{ab}T_{ab} \;&= (T^{rr})^2 + 2r^2(T^{r\theta})^2 + 2r^2\sin^2\theta(T^{r\phi})^2 \\
&\quad + r^4(T^{\theta\theta})^2 + 2r^4\sin^2\theta(T^{\theta\phi})^2 + r^4\sin^4\theta(T^{\phi\phi})^2 \\
\\
&= \;\sin^2\theta(\partial_\theta V)^2 + 2r^2\sin^2\theta(\partial_r V)^2 + 2r^2\sin^2\theta(\partial_\theta W)^2 \\
&\quad + r^{10}\left(\int\left[\frac{\sin\theta}{r^3}\partial_{rr}V - \frac{\cos\theta}{r^5}\partial_\theta V\right]d\theta\right)^2 + 2r^4\sin^2\theta(\partial_r W)^2 \\
&\quad + r^4\sin^2\theta\left(\frac{1}{r^2}\;\partial_\theta V + \frac{r}{\sin\theta}\int\left[\frac{\sin\theta}{r^3}\partial_{rr}V - \frac{\cos\theta}{r^5}\partial_\theta V\right]d\theta\right)^2 \; ,
\end{aligned} \tag{3.6.4}$$

involving complicated integrals, with many interconnected θ terms, making it difficult to chose V and W to remove the θ dependence.

3.6.1 Time-Rotation Symmetry

Assuming the "time-rotation" symmetry of *Brandt & Seidel* [12], see equation (3.1.7), vastly simplifies (3.6.4) by setting the potential V to zero, giving the product to be:

$$T^{ab}T_{ab} \;=\; 2r^2\sin^2\theta\left((\partial_\theta W)^2 + r^2(\partial_r W)^2\right) \;, \tag{3.6.5}$$

now a first order, non-linear, partial differential equation in W.

A simplified solution to (3.6.5) can be given by choosing the potential W to have *no* r dependence, therefore giving $\partial_r W = 0$. The partial derivative with respect to θ is then required to cancel with the $\sin^2\theta$, which can be integrated to give an expression for the potential W:

$$W \;=\; k\;\ln\left(Tan\left(\frac{\theta}{2}\right)\right) \qquad \Rightarrow \qquad \begin{aligned} T^{ab}T_{ab} &= 2r^2\sin^2\theta\left((\partial_\theta W)^2 + r^2(\partial_r W)^2\right) \\ &= 2r^2\sin^2\theta\left(k^2\sin^{-2}\theta + r^2(\cancel{\partial_r W})^2\right) \\ &= 2k^2r^2 \;, \end{aligned} \tag{3.6.6}$$

for an arbitrary constant k. This particular solution, however, leads to the product becoming unboundedly large as r increases. Any solution avoiding this must have a dependence on r, in the choice of the potential W.

Chapter 4

Kerr and Bowen-York Extrinsic Curvatures

4.1 Kerr and Bowen-York Background

The difference between the space-times formed by the Bowen-York initial data, introduced in section 2.2.5, and the Kerr space-time, described in section 1.3.4, have been investigated extensively in the literature. Some of the more influential of these works are outlined below.

4.1.1 *Bowen & York*

The Bowen-York initial data was first compared with the Kerr spacetime solution in the original Bowen & York paper [10], where it was stated that on the "time rotation" symmetry slice, see equation (3.1.7), of the Kerr metric in Boyer-Lindquist form, the extrinsic curvature agrees with the Bowen-York conformal curvature to order $O(r^{-3})$.

It was also noted that the constant time slices of the Kerr metric, in Boyer-Lindquist form, are not conformally flat, and hence the space-time evolved from the Bowen-York data will not be identical to the Kerr space-time. It was therefore concluded in [10], that the Bowen-York initial data would emit some "gravitational radiation", with the flux of any wave energy being zero on the initial slice, and then settle down to a Kerr space-time.

4.1.2 *Brandt & Seidel*

In the paper discussed briefly in section 3.1.2, *Brandt & Seidel* [12], a family of initial data sets for rotating black holes is constructed, with both Kerr and Bowen-York as special cases.

The spatial metric is given as an adjustment to the Brill wave metric from *Brill* [13], giving the spatial line element:

$$dl^2 \;=\; \psi^4 \left[e^{2(q-q_0)}(d\eta^2 + d\theta^2) + \sin^2\theta \; d\phi^2 \right] \;, \tag{4.1.1}$$

with the differential function q as a freely specifiable variable for the family of data.

Setting $q = q_0$, can easily be seen to give a conformally flat 3-metric, in line with the Bowen-York initial data. Setting $q = 0$, with the conformal factor ψ defined by equation (4a) of [12], then gives a spatial metric for the Kerr space-time, after a minor coordinate transformation from the Boyer-Lindquist coordinates, given by [12] (equations (2a) & (2b)).

As discussed in section 3.1.2, the "time-rotation" symmetry (3.1.7), is used to give the only non-zero part of the momentum constraint by (3.1.9):

$$\partial_\eta \hat{H}_E \ \sin^3\theta + \partial_\theta \left(\hat{H}_F \ \sin^2\theta \right) \ = \ 0 \ , \tag{4.1.2}$$

with $\hat{H}_E$ and $\hat{H}_F$ corresponding to the $\bar{K}_{\eta\phi}$ and $\bar{K}_{\theta\phi}$ components of the conformal curvature.

Distorted versions of the Kerr and Bowen-York initial data are then constructed in [12], by slight adjustments of the q term in the spatial metric (4.1.1).

4.1.3 *Gleiser, Nicasio, Price & Pullin*

The main goal of numerical relativity at present, is to provide information about the gravitational waves predicted to emanate from the interactions of massive objects. However, the "relaxation" of the Bowen-York initial data to a Kerr space-time, is expected to occur by the emission of "non-physical" gravitational radiation. See *Bowen & York* [10], described in section 4.1.1, and notice also from *Brandt & Seidel*, that the difference between the spatial metrics of Kerr and Bowen-York in equation (4.1.1), is given by a Brill wave.

As a result of the undesirable influence of this radiation, *Gleiser et al.* [28] attempt to compute the radiation emitted by the "relaxation" of Bowen-York initial data to Kerr black holes, using perturbations of Schwarzschild space-time.

Though the results produced by [28] are only considered to be relevant for rotations far below the extreme case ($J = M^2$), they show that the wave form is dominated by a "quasinormal ringing", and suspect that the initial burst "can contaminate the evolution for some time". This suggests that the difference between the Bowen-York initial data, and the Kerr space-time that it eventually evolves to, is too great for effective gravitational wave simulations.

4.1.4 *Garat & Price*

Although it can easily be shown that the Kerr metric in *Boyer-Lindquist* coordinates does not admit a conformally flat spatial metric, see for example *Bowen & York* [10] described in section 4.1.1, this does not necessarily imply that other coordinate systems cannot give a conformally flat spatial metric, for the Kerr space-time.

It is shown, however, in *Garat & Price* [26], that there are no conformally flat axially symmetric slices of the Kerr metric, which smoothly reduce to a slice of constant Schwarzschild time, in the Schwarzschild limit.

The condition of conformal flatness is found by evaluation of the Bach (Cotton-York) tensor, which is zero for any conformally flat metric. The tensor is then evaluated for an expansion of a Boyer-Lindquist time-slice, in the rotation factor a. These expansions will of course, smoothly reduce to constant time slices of the Schwarzschild space-time, as $a \to 0$.

4.1.5 *Dain, Lousto & Takahashi*

In *Dain et al.* [21], a set of initial conditions is constructed with a conformally flat spatial metric, and an extrinsic curvature closer to that of the Kerr metric, in Boyer-Lindquist coordinates.

The extrinsic curvature is constructed from the transverse trace-free tensor from *Dain* [19], described in section 3.1.3, and given by equation (3.1.15):

$$K^{ab} \;=\; \epsilon^{ajk}\,\frac{\eta_j \eta^b}{\eta^2}\,D_k\omega \;+\; \epsilon^{bjk}\,\frac{\eta_j \eta^a}{\eta^2}\,D_k\omega\;, \tag{4.1.3}$$

with the Killing vector η^a taken to be axially-symmetric, see equations (3.2.6) and (3.2.7). The choice for ω is then made to "resemble" the extrinsic curvature of the Kerr space-time, in Boyer Lindquist coordinates, and is given as an adjustment to the ω required to give the Bowen-York conformal curvature:

$$\omega_K \;=\; \omega_{BY} - \frac{Ma^3 \sin^4\theta\ \cos\theta}{\Sigma}\;, \tag{4.1.4}$$

with ω_{BY} given by W from equation (3.4.8), with $J = Ma$. Substituting (4.1.4) into equation (4.1.3) is shown by equation (27) of [21], to give the curvature components:

$$K_{r\phi} \;=\; \frac{Ma\left[(r^2-a^2)\Sigma + 2r^2(r^2+a^2)\right]}{r^2\Sigma^2}\ \sin^2\theta\;, \tag{4.1.5a}$$

$$K_{\theta\phi} \;=\; -\ \frac{2Ma^3 r}{\Sigma}\ \left(1 - \frac{M^2-a^2}{4r^2}\right)\ \cos\theta\ \sin^3\theta\;, \tag{4.1.5b}$$

with the remaining components vanishing.

The new data is shown to allow black holes of higher rotations than Bowen-York, and also to lead to less "spurious" radiation, while keeping the main advantage of the Bowen-York data, in the conformal flatness of the spatial metric.

4.2 Extrinsic Curvatures and Derivatives

For a comparison of the extrinsic curvatures of Kerr and Bowen-York, the extrinsic curvatures for each need to be given in a comparable coordinate system.

The extrinsic curvature for a Boyer-Lindquist time slice of the Kerr metric is calculated in the appendix section A.1, with the results given below in section 4.2.1. The coordinate derivatives of the curvature components are then calculated in the appendix section A.2, with the results given in section 4.2.2.

Finally, the angular momentum part of the Bowen-York conformal curvature is calculated in spherical-polar coordinates in section 4.2.3.

4.2.1 Kerr Extrinsic Curvature

The non-zero components of the extrinsic curvature of the Kerr metric, are given by equations (A.1.32), in terms of the lapse α, as:

$$K_{r\phi} = -\frac{\alpha M a \sin^2\theta\ \left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\triangle}\ , \tag{4.2.1a}$$

$$K_{\theta\phi} = \frac{2rMa^3\alpha\sin^3\theta\ \cos\theta}{\Sigma^2}\ , \tag{4.2.1b}$$

which can be approximated for large values of r, from equations (A.1.33), by:

$$K_{r\phi} \simeq -\frac{3Ma\sin^2\theta}{r^2}\ , \tag{4.2.2a}$$

$$K_{\theta\phi} \simeq \frac{2Ma^3\sin^3\theta\ \cos\theta}{r^3}\ . \tag{4.2.2b}$$

With indices raised, the non-zero extrinsic curvature components are given by equations (A.1.36), in terms of the lapse α, as:

$$K^{\phi r} = \frac{-\alpha M a\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\ , \tag{4.2.3a}$$

$$K^{\phi\theta} = \frac{2rMa^3\alpha\sin\theta\ \cos\theta}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\ . \tag{4.2.3b}$$

In the limit as the r coordinate tends to infinity, the *raised* index curvature components are given by (A.1.37), as:

$$K^{r\phi} \simeq -\frac{3Ma}{r^4}, \tag{4.2.4a}$$

$$K^{\theta\phi} \simeq \frac{2Ma^3 \sin\theta \, \cos\theta}{r^7}. \tag{4.2.4b}$$

4.2.2 Derivatives of the Curvature Components of Kerr

The partial derivative of the curvature component $K^{r\phi}$, given by equation (4.2.3a), with respect to the coordinate r, is given from equation (A.2.9), in terms of the lapse α, by:

$$\begin{aligned} \partial_r \, K^{\phi r} = & \frac{\alpha M a}{\Sigma^2 \left((r^2+a^2)^2 - \triangle a^2 \sin^2\theta\right)} \\ & \Bigg(-2r\left(6r^2 + a^2 + a^2\cos^2\theta\right) \\ & + \frac{(2r^3 - 5r^2 M + 3ra^2 - ra^2\cos^2\theta + Ma^2\cos^2\theta)}{\Sigma\triangle} \\ & \left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right) \\ & + \frac{3\left(2r^3 + ra^2 + ra^2\cos^2\theta + Ma^2\sin^2\theta\right)}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\ & \left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right) \Bigg). \end{aligned} \tag{4.2.5}$$

Similarly, the partial derivative of the curvature component $K^{\theta\phi}$, given by equation (4.2.3b), with respect to the coordinate θ, is given from equation (A.2.14), in terms of the lapse α, by:

$$\begin{aligned} \partial_\theta \, K^{\phi\theta} = & \frac{2r\alpha M a^3}{\Sigma^2 \left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\ & \left(\cos^2\theta - \sin^2\theta + \frac{3a^2\cos^2\theta \, \sin^2\theta}{\Sigma} + \frac{3a^2\triangle\cos^2\theta \, \sin^2\theta}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\right). \end{aligned} \tag{4.2.6}$$

The simplifications of the Boyer-Lindquist coordinates Σ and $\triangle$, for all of the equations in sections 4.2.1 and 4.2.2, are given from equation (1.3.35), by:

$$\begin{aligned} \triangle &= r^2 - 2Mr + a^2 \ , \\ \Sigma &= r^2 + a^2 \cos^2\theta \ , \end{aligned} \tag{4.2.7}$$

with the angular momentum given by $J = Ma$, for the mass M and rotation factor a. The lapse α can be calculated from equation (A.1.16).

4.2.3 Bowen-York Curvature

The Bowen-York conformal extrinsic curvature, with angular momentum J^a, is given by equation (2.2.61):

$$\bar{K}_{ab} = \frac{3}{r^3} \ (\epsilon_{acd} \ q_b + \epsilon_{bcd} \ q_a) \, q^c J^d \ , \tag{4.2.8}$$

with q^a the unit normal of a sphere of constant radius, and ϵ_{abc} the Levi-Civita alternating tensor described by equation (2.2.57).

In *spherical* coordinates $(r, \ \theta, \ \phi)$, with the angular momentum directed in the direction of the zenith, the unit space-like normal q^a and angular momentum vector are given by:

$$q^a = q_a = (1, \ 0, \ 0) \ , \qquad J^a = J \left(\cos\theta, -\frac{1}{r}\sin\theta, 0 \right) \ . \tag{4.2.9}$$

The Bowen-York conformal curvature is then given in cylindrical coordinates by:

$$\begin{aligned} \bar{K}_{ab} &= \frac{3}{r^3} \ (\epsilon_{acd} \ q_b + \epsilon_{bcd} \ q_a) \, q^c J^d \\ &= \frac{3}{r^3} \ (\epsilon_{ar\theta} \ q_b + \epsilon_{br\theta} \ q_a) \, q^r J^\theta \ , \end{aligned} \tag{4.2.10}$$

since, for $\bar{K}_{ab}$ to be non-zero $q^i = q^r$, and hence by the alternating tensor, $J^i = J^\theta$. Also, since the only non-zero choice left for ϵ is $a, b = \phi$, and again the only non-zero choice for q_i gives $a, b = r$, the only non-zero components of $\bar{K}_{ab}$ must be given by:

$$\begin{aligned} \bar{K}_{r\phi} = \bar{K}_{\phi r} &= \frac{3}{r^3} \ \epsilon_{\phi r\theta} \ q_r \ q^r \ J^\theta \\ &= \frac{3}{r^3} \ (\sqrt{\gamma}) \ (1) \ (1) \ \left(-\frac{J\sin\theta}{r} \right) \\ &= \frac{3(r^2\sin\theta)}{r^3} \ \left(-\frac{J\sin\theta}{r} \right) \\ &= - \ \frac{3J\sin^2\theta}{r^2} \ . \end{aligned} \tag{4.2.11}$$

In the Kerr metric, the angular momentum is given in terms of a rotational factor a such that $J = Ma$, with M reducing to the Schwarzschild mass, in the limit $a \to 0$. Hence, in terms of M and a, the non-zero components of the Bowen-York conformal extrinsic curvature are given by:

$$\bar{K}_{r\phi} \;=\; \bar{K}_{\phi r} \;=\; -\frac{3Ma\sin^2\theta}{r^2} \; . \tag{4.2.12}$$

The non-zero components of the conformal curvature, with raised indices, are then given by:

$$\bar{K}^{r\phi} \;=\; \bar{K}^{\phi r} \;=\; -\frac{3Ma}{r^4} \; . \tag{4.2.13}$$

The conformal factor is found by solving the Hamiltonian constraint:

$$8\bar{D}^2\psi - \cancelto{0}{\bar{R}}\psi + \psi^7\bar{A}_{ab}\bar{A}^{ab} - \frac{2}{3}\psi^5\cancelto{0}{K^2} + 16\pi\psi^5\cancelto{0}{\rho} \;=\; 0 \; ,$$

$$\Leftrightarrow \qquad 8\bar{D}^2\psi + \psi^7\bar{A}_{ab}\bar{A}^{ab} \;=\; 0 \; ,$$

$$\Leftrightarrow \qquad \bar{D}^2\psi + \psi^7\left(\frac{9M^2a^2\sin^2\theta}{4r^6}\right) \;=\; 0 \; , \tag{4.2.14}$$

$$\tag{4.2.15}$$

though unfortunately this cannot be solved exactly, requiring numerical techniques, and the setting of two boundary conditions.

4.3 Comparing the Curvatures of Kerr and Bowen-York

In this section, a number of basic calculations are carried out, to show some of the differences between the extrinsic curvature tensors of the Boyer-Lindquist time-slice of the Kerr metric, and the angular momentum part of the Bowen-York initial data.

4.3.1 Tensor Difference

Giving the extrinsic curvature tensor for the Kerr metric, in terms of the Bowen-York curvature, involves the addition of an extra tensor denoted by k^{ab}:

$$K_K^{ab} \;=\; \psi^{-10}\bar{K}_{BY}^{ab} \;+\; k^{ab} \;. \tag{4.3.1}$$

Giving K_K^{ab} from equations (4.2.3), and $\bar{K}_{BY}^{ab}$ by equation (4.2.13), the non-zero components of the difference k^{ab} are given by:

$$\begin{aligned} k^{ab} &= K_K^{ab} - \psi^{-10}\bar{K}_{BY}^{ab} \;, \\ \Leftrightarrow \qquad k^{r\phi} \;=\; k^{\phi r} &= -\frac{\alpha M a\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\ &\quad + \psi^{-10}\,\frac{3Ma}{r^4}\;, \end{aligned} \tag{4.3.2a}$$

$$k^{\theta\phi} \;=\; k^{\phi\theta} \;=\; \frac{2rMa^3\alpha\sin\theta\;\cos\theta}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\;. \tag{4.3.2b}$$

However, the conformal factor ψ cannot be given without solving the second order elliptic equation (4.2.14).

The difficulty with the conformal factor can be avoided, by calculating k^{ab} close to one of the boundary conditions, where an expression for the conformal factor can be given with greater certainty. Since the Bowen-York space-time is expected to be assymptotically flat, as $r \to \infty$ the conformal factor $\psi \to 1$. Hence, at large values of r, the components of K_K^{ab} can be approximated by equations (4.2.4), and the Bowen-York conformal factor by 1.

From equation (4.3.1), the tensor k^{ab} for large values of r, is given by:

$$r \to \infty \qquad \Rightarrow \qquad k^{ab} \ \simeq \ K^{ab}_{K} - \bar{K}^{ab}_{BY} \ . \tag{4.3.3}$$

and substituting from equations (4.2.4) and (4.2.13), into (4.3.3), the non-zero components of the difference k^{ab}, for large values of r, are given by:

$$\begin{aligned} r \to \infty \qquad \Rightarrow \qquad k^{r\phi} \ = \ k^{\phi r} \ &\simeq K^{r\phi}_{K} - \bar{K}^{r\phi}_{BY} \\ &\simeq \ -\frac{3Ma}{r^4} + \frac{3Ma}{r^4} \\ &= 0 \ , \end{aligned} \tag{4.3.4a}$$

$$\begin{aligned} k^{\theta\phi} \ = \ k^{\phi\theta} \ &\simeq K^{\theta\phi} \\ &\simeq \frac{2Ma^3 \sin\theta \ \cos\theta}{r^7} \ , \end{aligned} \tag{4.3.4b}$$

With indices *lowered*, the only non-zero components of k_{ab}, can be seen for large r to be:

$$\begin{aligned} r \to \infty \qquad \Rightarrow \qquad k_{\theta\phi} \ = \ k_{\phi\theta} \ &\simeq K_{\theta\phi} \\ &\simeq \frac{2Ma^3 \sin^3\theta \ \cos\theta}{r^3} \ . \end{aligned} \tag{4.3.5}$$

Calculating the contraction of both upper and lower index tensors:

$$r \to \infty \qquad \Rightarrow \qquad \left(k_{ab}k^{ab}\right) \ \simeq \ 2\left(k_{\theta\phi}k^{\theta\phi}\right) \ \simeq \ \frac{6M^2a^6 \sin^4\theta \ \cos^2\theta}{r^{10}} \ , \tag{4.3.6}$$

shows the extrinsic curvature tensors of Kerr and Bowen-York, to differ on the order of r^{-5}, for large values of r.

The scalar products of each curvature can be seen from equations (4.2.14) and (A.3.7), to both be given by:

$$r \to \infty \qquad \Rightarrow \qquad \left(K_{ab}K^{ab}\right)_{BY} \ \simeq \ \left(K_{ab}K^{ab}\right)_{K} \ \simeq \ \frac{18M^2a^2 \sin^2\theta}{r^6} \ , \tag{4.3.7}$$

showing the Kerr and Bowen-York curvatures to be equivalent to order r^{-3}, agreeing with *Bowen & York* [10], as seen in section 4.1.1.

4.3.2 Flat Divergence of Tensor Difference

Both Kerr extrinsic curvature and Bowen-York conformal curvature tensors can easily be seen to be trace-free, and each are transverse with respect to the their spatial metric and conformal spatial metric respectively. Hence, it may prove useful to find how close the Kerr curvature is to being divergence-free with respect to the flat metric.

The connection coefficients for the flat 3-metric, in spherical polar coordinates, were given in equation (3.2.3), by:

$$\begin{aligned} \Gamma^r_{\theta\theta} &= -r \,, & \Gamma^\theta_{r\theta} &= \Gamma^\theta_{\theta r} = \frac{1}{r} \,, & \Gamma^\phi_{r\phi} &= \Gamma^\phi_{\phi r} = \frac{1}{r} \,, \\ \Gamma^r_{\phi\phi} &= -r\ \sin^2\theta \,, & \Gamma^\theta_{\phi\phi} &= -\cos\theta\ \sin\theta \,, & \Gamma^\phi_{\theta\phi} &= \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta} \,. \end{aligned} \tag{4.3.8}$$

With respect to the flat 3-metric, the divergence of the Kerr extrinsic curvature is then given by:

$$D_b K^{ab}_K = \partial_b K^{ab}_K + \Gamma^a_{bc} K^{cb} + \Gamma^b_{bc} K^{ac} \,, \tag{4.3.9}$$

and separated into its individual components:

$$\begin{aligned} D_b K^{rb}_K &= \partial_b K^{rb}_K + \Gamma^r_{bc} K^{cb}_K + \Gamma^b_{bc} K^{rc}_K \\ &= \cancel{\partial_\phi K^{r\phi}_K} + \Gamma^r_{\theta\theta} \cancel{K^{\theta\theta}_K} + \Gamma^r_{\phi\phi} \cancel{K^{\phi\phi}_K} + \cancel{\Gamma^b_{b\phi}} K^{r\phi}_K \\ &= 0 \,, \end{aligned} \tag{4.3.10a}$$

$$\begin{aligned} D_b K^{\theta b}_K &= \partial_b K^{\theta b}_K + \Gamma^\theta_{bc} K^{cb}_K + \Gamma^b_{bc} K^{\theta c}_K \\ &= \cancel{\partial_\phi K^{\theta\phi}_K} + 2\ \Gamma^\theta_{r\theta} \cancel{K^{r\theta}_K} + \Gamma^\theta_{\phi\phi} \cancel{K^{\phi\phi}_K} + \cancel{\Gamma^b_{b\phi}} K^{\theta\phi}_K \\ &= 0 \,, \end{aligned} \tag{4.3.10b}$$

$$\begin{aligned} D_b K^{\phi b}_K &= \partial_b K^{\phi b}_K + \Gamma^\phi_{bc} K^{cb}_K + \Gamma^b_{bc} K^{\phi c}_K \\ &= \partial_r K^{\phi r}_K + \partial_\theta K^{\phi\theta}_K + 2\ \Gamma^\phi_{r\phi} K^{r\phi}_K + 2\ \Gamma^\phi_{\theta\phi} K^{\theta\phi}_K \\ &\quad + \Gamma^\theta_{\theta r} K^{\phi r}_K + \Gamma^\phi_{\phi r} K^{\phi r}_K + \Gamma^\phi_{\phi\theta} K^{\phi\theta}_K \\ &= \partial_r K^{\phi r}_K + \partial_\theta K^{\phi\theta}_K + \frac{4}{r}\ K^{\phi r}_K + \frac{3\ \cos\theta}{\sin\theta}\ K^{\phi\theta}_K \,, \end{aligned} \tag{4.3.10c}$$

expectedly giving the ϕ component as the only non-zero part of the divergence.

The derivatives of the curvature components, given by equations (4.2.5) and (4.2.6), and the curvature components themselves, from equations (4.2.3a) and (4.2.3b), can be substituted into the ϕ part of the divergence (4.3.10c), giving:

$$
\begin{aligned}
D_b K_K^{\phi b} &= \partial_r K_K^{\phi r} + \partial_\theta K_K^{\phi\theta} + \frac{4}{r}\, K_K^{\phi r} + \frac{3\, \cos\theta}{\sin\theta}\, K_K^{\phi\theta} \\
&= \frac{\alpha M a}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2 \sin^2\theta\right)} \\
&\quad \Bigg(-2r\left(6r^2 + a^2 + a^2\cos^2\theta\right) \\
&\quad + \frac{(2r^3 - 5r^2 M + 3ra^2 - ra^2\cos^2\theta + Ma^2\cos^2\theta)}{\Sigma\triangle} \\
&\quad \left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right) \\
&\quad + \frac{3\left(2r^3 + ra^2 + ra^2\cos^2\theta + Ma^2\sin^2\theta\right)}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&\quad \left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right) \Bigg) \\
&\quad + \frac{2r\alpha M a^3}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&\quad \left(\cos^2\theta - \sin^2\theta \;+\; \frac{3a^2\cos^2\theta\, \sin^2\theta}{\Sigma} \;+\; \frac{3a^2\triangle\cos^2\theta\, \sin^2\theta}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \right) \\
&\quad + \frac{4}{r}\, \frac{-\alpha M a\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&\quad + \frac{3\, \cos\theta}{\sin\theta}\, \frac{2rMa^3\alpha\sin\theta\, \cos\theta}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\ ,
\end{aligned}
\tag{4.3.11}
$$

with α representing the lapse for the Kerr metric, which can be evaluated explicitly by use of equation (A.1.16), and equations (A.1.3) for the Kerr space-time metric components, in Boyer-Lindquist form.

For large values of r again, equation (4.3.11) reduces to:

$$
\begin{aligned}
D_b K_K^{\phi b} &= \partial_r K_K^{\phi r} + \partial_\theta K_K^{\phi\theta} + \frac{4}{r}\ K_K^{\phi r} + \frac{3\ \cos\theta}{\sin\theta}\ K_K^{\phi\theta} \\
&= \frac{12Ma}{r^5} \\
&\quad + \frac{2Ma^3\left(\cos^2\theta - \sin^2\theta\right)}{r^7} \\
&\quad - \frac{4}{r}\ \frac{3Ma}{r^4} \\
&\quad + \frac{3\ \cos\theta}{\sin\theta}\ \frac{2Ma^3\sin\theta\ \cos\theta}{r^7} \\
&= \cancel{\frac{12Ma}{r^5}} + \frac{2Ma^3\left(\cos^2\theta - \sin^2\theta\right)}{r^7} - \cancel{\frac{12Ma}{r^5}} + \frac{6Ma^3\cos^2\theta}{r^7} \\
&= -\ \frac{2Ma^3\left(1+\cos^2\theta\right)}{r^7}\ .
\end{aligned}
\tag{4.3.12}
$$

The flat divergence can also be found for the difference tensor k^{ab}, by applying the divergence to equation (4.3.1):

$$
\begin{aligned}
k^{ab} &= K_K^{ab} - \psi^{-10}\bar{K}_{BY}^{ab}\ , \\
\bar{D}_b k^{ab} &= \bar{D}_b K_K^{ab} - \bar{K}_{BY}^{ab}\ \bar{D}_b(\psi^{-10})\ .
\end{aligned}
\tag{4.3.13}
$$

From equations (4.3.11) and (4.2.13), the only non-zero component is given by:

$$
\bar{D}_b k^{\phi b} \ = \ \bar{D}_b K_K^{\phi b} - \frac{30Ma}{r^4}\ \psi^{-11}\partial_r\psi\ , \tag{4.3.14}
$$

with the $\bar{D}_b K_K^{\phi b}$ term given by equation (4.3.11). For large r, the conformal factor can be given by 1, and equation (4.3.12) used, giving:

$$
\lim_{r\to\infty}\bar{D}_b k^{\phi b} \ = \ \lim_{r\to\infty}\bar{D}_b K_K^{\phi b} = \ -\frac{2Ma^3\left(1+\cos^2\theta\right)}{r^7}\ , \tag{4.3.15}
$$

which is of the same order as the tensor k^{ab} itself, from equation (4.3.4).

Chapter 5

Brill Wave Mass-Energy Approximations

5.1 Brill Wave Mass Evaluation

5.1.1 *Dieter Brill*

Until 1959, it was not certain whether gravitational waves could be considered to contain "mass". However, *Brill* [13] proved a positive-definiteness theorem for the mass of gravitational waves, showing the mass to be non-negative, and to vanish only in the case of a flat metric.

Brill Wave Metric

The space-times examined in *Brill* [13], are assumed to be source-free, axially-symmetric, and to have a time-symmetry for a particular time coordinate. This time-symmetry can be seen as the "implosion" of a wave about an axis, before it "re-emits". The "moment of time-symmetry" then forms a space-like hypersurface, with a vanishing 3-scalar curvature.

For a source-free space-time, the Hamiltonian constraint (2.1.27) shows that the condition ${}^{(3)}R = 0$, implies a zero extrinsic curvature tensor K^{ab}, though this form of the constraint equations came much later.

In order to have a well-defined concept of an invariant "mass" of the wave, the space-time is also assumed to be asymptotically Schwarzschild, and hence, asymptotically flat. What has become known as the "Brill wave metric", was credited in *Brill* [13], to H. Bondi, and given in equation (3) of [13], in line element form, by:

$$ds^2 = e^{2p}\left[e^{2q}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right] . \tag{5.1.1}$$

The form of the metric used in this thesis, separates a constant function A from the differential function q, as in equation (9) of *Cantor & Brill* [14], and gives the conformal factor in the form of equation (19) of [13]:

$$ds^2 = \psi^4\left[e^{2Aq}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right] . \tag{5.1.2}$$

The conformal part of the metric here is equivalent to the form used in section 3.3, and the conformal factor ψ follows the notation introduced in section 2.2.1.

Both the conformal factor, and the differential function $q(\rho, z)$ in equation (5.1.2) are required to satisfy certain continuity conditions on the axis of symmetry. From equation (21) of [13], these are given by:

$$q \,\Big|_{\rho=0} \;=\; \partial_\rho q \,\Big|_{\rho=0} \;=\; \partial_\rho \psi \,\Big|_{\rho=0} \;=\; 0 \; . \tag{5.1.3}$$

The condition of axial-symmetry also imposes conditions on ψ and q:

$$\partial_\phi \psi \;=\; \partial_\phi q \;=\; 0 \; , \tag{5.1.4}$$

as can be seen from equation (3.2.7):

Positivity of Brill Wave Mass

It can be seen that the r^{-1} term in a multipole expansion of the spatial metric, should give the mass of the system. This term can then be integrated over a closed 2-surface, enclosing any "sources" of mass-energy.

For appropriate boundary conditions on q, which give an asymptotically flat metric, the multipole expansion of the conformal metric in equation (5.1.2), can be seen from [13], equations (23)-(25), to contain no r^{-1} term. The mass term of the gravitational wave must therefore be contained in the conformal factor of equation (5.1.2).

The conformal factor is found by solving the Lichnerowicz equation, or the conformal Hamiltonian constraint given by equation (2.2.7), which reduces to:

$$8 \; \bar{D}^2 \psi - \psi \; \bar{R} = \; 0 \; , \tag{5.1.5}$$

and can be seen by equation (2.2.6) to be given by ${}^{(3)}R = 0$, which is simply the time-symmetry condition of the hypersurface.

With the conformal factor, in the limit $r \to \infty$, assumed to be of the form:

$$\psi \;=\; 1 + \frac{M}{2r} + O\left(r^{-2}\right) \; , \tag{5.1.6}$$

from equation (22), equation (32) of [13] gives the mass of the gravitational wave by:

$$M \;=\; \frac{1}{2\pi} \int_\infty \left(D_F \ln \psi\right)^2 \; dV \; , \tag{5.1.7}$$

with D_F representing the flat 3-space covariant derivative, and dV, the flat space volume element. Since ψ must be everywhere positive, (5.1.7) shows the mass of the gravitational wave to be everywhere positive.

Scattering Solution for Conformal Factor

Brill continues in [13], by making an analogy for the elliptic equation of the conformal factor. Defining a new variable Φ, by:

$$\Phi \;=\; \frac{1}{4}\Big(\partial_{\rho\rho}(Aq) + \partial_{zz}(Aq)\Big) \;=\; \frac{1}{4}A\left(\partial_{\rho\rho}q + \partial_{zz}q\right) \;, \tag{5.1.8}$$

equation (5.1.5) can be given by:

$$\Big(D_F^2 + \Phi\Big)\,\psi \;=\; 0 \;, \tag{5.1.9}$$

with D_F^2 denoting the flat space Laplacian. This equation is of the same form as a 3-dimensional Schrödinger equation. In this sense, "ψ would describe the scattering of a unit density of particles of zero energy on the potential $-\Phi$."

In *Ó Murchadha* [41], it is explained that John Wheeler, in *Wheeler* [50], took a spherically symmetric square well potential in flat space, given by:

$$\begin{aligned} u \;&=\; -B \;, \qquad && r \;\leq\; a \\ u \;&=\; 0 \;, && r \;>\; a \;, \end{aligned} \tag{5.1.10}$$

and solved the equation:

$$D^2\psi - u\psi \;=\; 0 \;, \qquad \lim_{r\to\infty} \psi \;=\; 1 \;, \tag{5.1.11}$$

giving the solution, outside the potential, as:

$$\psi \;=\; 1 + \frac{a}{r}\left(\frac{Tan(B^{1/2}a)}{B^{1/2}a} - 1\right) \;. \tag{5.1.12}$$

The coefficient of the r^{-1} term in equation (5.1.12) can be seen as equivalent to the mass in equation (5.1.6). The graph of a term similar to this can be seen in figure 5.1, but note that for the Brill wave, the existence of a time-symmetry slice breaks down before the first asymptote is reached, see section 3 of *Cantor & Brill* [14].

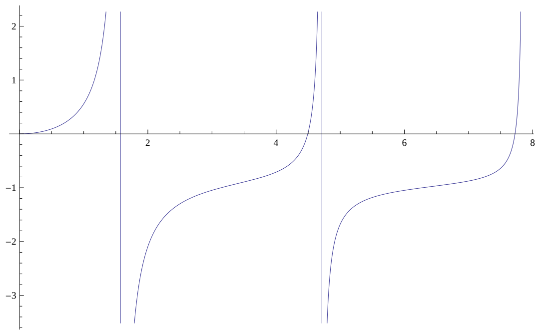


Figure 5.1: Graph of $\left(\frac{Tan\theta}{\theta} - 1\right)$ for θ between 0 and $5\pi/3$.

5.1.2 Solving the Conformal Factor

Creating a set of initial conditions for the Brill gravitational radiation involves setting the differential function given by $A\,q(\rho, z)$, and then solving equation (5.1.5) for the conformal factor. A number of different choices have been made for Aq, however the partial differential equation for the conformal factor must be solved numerically.

Eppley

One of the first numerical solutions giving Brill wave initial conditions was given in *Eppley* [24], in 1977. The conformal metric was determined, in equation (5) of [24], by the choice:

$$A\,q \;=\; \frac{A\,\rho^2}{1 + (r/\lambda)^n}\;, \tag{5.1.13}$$

with A and λ constants, $r^2 = \rho^2 + z^2$, and $n \geq 4$.

As well as forming a set of initial conditions, *Eppley* [24] also investigated the effect of the metric "amplitude" A, on the mass of the wave. It was shown, in agreement with the scattering problem analogy of Wheeler, that the total mass $M \propto A^2$, for small A.

It was also shown, that increasing the amplitude would eventually lead to the formation of a black hole, with the "sudden appearance" of an apparent horizon. As A is increased further, the horizon enlarges all the way to spatial infinity, with the mass growing infinitely large. This can also be seen in figure 5.1, prior to the first asymptote.

Gentle, Holz, Miller & Wheeler

More recently, in 1998 *Gentle et al.* [27] used two different numerical techniques, to produce initial conditions for a class of Brill waves. The differential function for the conformal metric, is given by the choice:

$$A\,q \;=\; A\,\rho^2\,e^{-r^2}\;, \tag{5.1.14}$$

with $r^2 = \rho^2 + z^2$. The critical point of the amplitude A, for the appearance of a black hole in the initial conditions, is also investigated in [27].

5.2 Numerical Approximation of Brill Wave Mass

As seen in *Eppley* [24] and *Gentle et al.* [27], solving even the axially symmetric elliptic equations for the conformal factor, is not an easy task. In order to make some progress, however, an attempt is made in this section to reduce the complexity of the both the Hamiltonian equation, and the mass calculations.

5.2.1 Deriving a Mass Expression for a Function $q(\rho, z)$

In order to find the effect of the scalar value A on the mass of a Brill wave, a function $q(\rho, z)$ needs to be defined. The Lichnerowicz equation must then be given for this function $q(\rho, z)$, so that the conformal factor $\psi(\rho, z)$ can be solved for, and finally, an expression needs to be given for the mass-energy of the space-time.

Calculating the Conformal Factor

By direct calculation, using the connection coefficients of the Brill metric from equation (3.3.4), substituted into equation (1.2.70) for the Riemann curvature tensor, the conformal curvature of a general Brill metric is given by:

$$\bar{R} \;=\; -2Ae^{-2Aq}\left(\partial_{\rho\rho}q + \partial_{zz}q\right) \ . \tag{5.2.1}$$

Substituting into Lichnerowicz equation (5.1.5) gives:

$$\begin{aligned}
0 \;&= 8\ \bar{D}^2\psi - \psi\ \bar{R} \\
&= 8\ \bar{\gamma}^{ab}\bar{D}_a\bar{D}_b\psi + 2Ae^{-2Aq}\left(\partial_{\rho\rho}q + \partial_{zz}q\right)\ \psi \\
&= e^{-2Aq}\partial_\rho\partial_\rho\psi + e^{-2Aq}\partial_z\partial_z\psi - \gamma^{aa}\Gamma^c_{aa}\partial_c\psi + \frac{1}{4}Ae^{-2Aq}\left(\partial_{\rho\rho}q + \partial_{zz}q\right)\ \psi \\
&= e^{-2Aq}\partial_{\rho\rho}\psi + e^{-2Aq}\partial_{zz}\psi + \frac{1}{\rho}e^{-2Aq}\partial_\rho\psi + \frac{1}{4}Ae^{-2Aq}\left(\partial_{\rho\rho}q + \partial_{zz}q\right)\ \psi \\
&= \partial_{\rho\rho}\psi + \partial_{zz}\psi + \frac{1}{\rho}\partial_\rho\psi + \frac{1}{4}A\left(\partial_{\rho\rho}q + \partial_{zz}q\right)\ \psi \ ,
\end{aligned} \tag{5.2.2}$$

using again the connection coefficients from equation (3.3.4). From the definition of Φ given in (5.1.8), equation (5.2.2) can be given by:

$$D_F^2\psi + \Phi\ \psi \;=\; 0 \ , \tag{5.2.3}$$

with D_F^2 again denoting the flat space Laplacian, which can easily be seen as equivalent to (5.2.2).

Calculating the Mass-Energy

From section 5.1.1, it can be seen that the mass-energy of a Brill space-time is contained in the conformal factor ψ. It can also be seen from *Brill* [13], that this is given by the expression:

$$M = -\oint_{\infty} \bar{D}\psi \, dS \ , \tag{5.2.4}$$

for some 2-surface S, evaluated at an infinite distance.

Using Stokes theorem, this can be transformed into a volume integral:

$$\begin{aligned} M &= -\frac{1}{2\pi} \oint_{\infty} \bar{D}\psi \, dS \\ &= -\frac{1}{2\pi} \int_{\infty} \bar{D}^2\psi \, dv \\ &= -\frac{1}{2\pi} \int_{\infty} \frac{1}{8}\bar{R}\psi \, dv \\ &= \frac{1}{2\pi} \int_{\infty} e^{-2Aq} \, \Phi\psi \, \rho e^{2Aq} \, \mathrm{d}\rho \, dz \, d\phi \\ &= \int_0^{\infty} \rho \left[\int_{-\infty}^{\infty} \Phi\psi \, dz \right] d\rho \ , \end{aligned} \tag{5.2.5}$$

also using the Lichnerowicz equation (5.1.5), and the relation between the conformal curvature $\bar{R}$ and the function Φ, by comparing equations (5.2.1) and (5.1.8).

5.2.2 Simplification of Lichnerowicz Equation

The choice of the function $q(\rho, z)$ is arbitrary, once equations (5.1.3) and (5.1.4) are satisfied, with two very different functions given in *Eppley* [24] and *Gentle et al.* [27], see equations (5.1.13) and (5.1.14).

Lichnerowicz equation for the conformal factor $\psi(\rho, z)$, given by the form (5.2.2):

$$\partial_{\rho\rho}\,\psi(\rho, z) + \partial_{zz}\,\psi(\rho, z) + \frac{1}{\rho}\,\partial_{\rho}\,\psi(\rho, z) + \frac{1}{4}A\left(\partial_{\rho\rho}q + \partial_{zz}q\right)\,\psi(\rho, z) \;=\; 0\;, \tag{5.2.6}$$

can be seen to reduce to an *ordinary* differential equation, if the following two conditions are satisfied:

$$\partial_{zz}\,\psi(\rho, z) \;= 0\;, \tag{5.2.7a}$$

$$\partial_{zz}\,q(\rho, z) \;= 0\;. \tag{5.2.7b}$$

Applying the technique of separation of variables to the conformal factor, such that $\psi(\rho, z) = \psi_{\rho}(\rho)\,\psi_{z}(z)$, unfortunately leads to solutions which are periodic about "0", if condition (5.2.7a) is not satisfied, and so do not satisfy the requirement for ψ to be positive everywhere.

Solutions to equations (5.2.7), which satisfy the requirement that ψ to be non-negative everywhere, are either constant in z, or tend to plus or minus infinity at some point. Taking the case where both functions are constant in z:

$$q(\rho, z) \;\equiv\; q(\rho)\;, \tag{5.2.8a}$$

$$\psi(\rho, z) \;\equiv\; \psi(\rho)\;, \tag{5.2.8b}$$

the mass-energy for the wave, given by equation (5.2.5), then becomes:

$$M \;=\; \int_0^\infty \rho\,\Phi(\rho)\,\psi(\rho)\,d\rho\, \int_{-\infty}^\infty dz\;. \tag{5.2.9}$$

The integral with respect to z is necessarily infinite, however, the integral with respect to ρ can give a "mass-factor":

$$M_\rho \;=\; \int_0^\infty \rho\,\Phi(\rho)\,\psi(\rho)\,d\rho\;, \tag{5.2.10}$$

with the proper mass M depending on this factor.

Choosing a Function $q(\rho)$

The function $q(\rho)$ is chosen to be similar to that of *Gentle et al.* [27], given earlier by equation (5.1.14), though with the z dependence removed:

$$q(\rho) \ := \ \rho^2 e^{-\rho^2} \ . \tag{5.2.11}$$

The second derivative of this function, with respect the ρ is given by:

$$\partial_{\rho\rho} q(\rho) \ = \ (4\rho^4 - 10\rho^2 + 2) \ e^{-\rho^2} \ , \tag{5.2.12}$$

giving the function $\Phi(\rho)$, from equation (5.1.8), by:

$$\begin{aligned} \Phi(\rho) \ &= \frac{1}{4} A \, (\partial_{\rho\rho} q) \\ &= \frac{1}{4} A (4\rho^4 - 10\rho^2 + 2) \ e^{-\rho^2} \\ &= \frac{1}{2} A (2\rho^4 - 5\rho^2 + 1) \ e^{-\rho^2} \ . \end{aligned} \tag{5.2.13}$$

The *ordinary* differential equation for the conformal factor $\psi(\rho)$ is then given, from equation (5.2.6), by:

$$\partial_{\rho\rho} \, \psi(\rho) + \frac{1}{\rho} \, \partial_\rho \, \psi(\rho) + \frac{1}{4} A \, (\partial_{\rho\rho} q) \ \psi(\rho) \ = \ 0 \ ,$$

$$\Leftrightarrow \qquad \partial_{\rho\rho} \, \psi(\rho) + \frac{1}{\rho} \, \partial_\rho \, \psi(\rho) + \frac{1}{2} A (2\rho^4 - 5\rho^2 + 1) \ e^{-\rho^2} \ \psi(\rho) \ = \ 0 \ , \tag{5.2.14}$$

and the mass factor M_ρ from equation (5.2.10), substituting for $\Phi(\rho)$ from equation (5.2.13), becomes:

$$\begin{aligned} M_\rho \ &= \int_0^\infty \rho \, \Phi(\rho) \, \psi(\rho) \, d\rho \\ &= \int_0^\infty \rho \, \frac{1}{2} A (2\rho^4 - 5\rho^2 + 1) \ e^{-\rho^2} \ \psi(\rho) \, d\rho \\ &= \frac{1}{2} A \int_0^\infty \rho \, (2\rho^4 - 5\rho^2 + 1) \ e^{-\rho^2} \ \psi(\rho) \, d\rho \ , \end{aligned} \tag{5.2.15}$$

requiring only the solution of the conformal factor from equation (5.2.14), to be given explicitly.

5.2.3 Numerical Approximations

Solving for the Conformal Factor

The conformal factor $\psi(\rho)$ was found using the "`NDSolve`" operation in *Mathematica*, on equation (5.2.14):

$$\partial_{\rho\rho}\,\psi(\rho) + \frac{1}{\rho}\,\partial_\rho\,\psi(\rho) + \frac{1}{2}A(2\rho^4 - 5\rho^2 + 1)\;e^{-\rho^2}\;\psi(\rho) \;=\; 0\;. \tag{5.2.16}$$

For the boundary conditions for ψ, the first was given to satisfy equation (5.1.3):

$$\partial_\rho\psi(\rho = 0) \;:=\; 0\;. \tag{5.2.17}$$

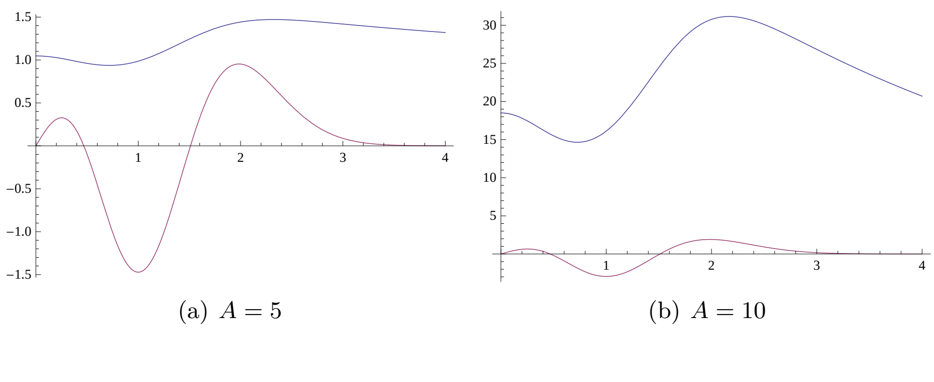
The second boundary condition was not so straight forward however, with a requirement for the conformal factor to act like:

$$\psi(\rho) \;=\; 1 + \frac{A}{\rho} + O(\rho^{-2})\;, \tag{5.2.18}$$

for large ρ and the requirement for ψ to be greater than or equal zero at all points. In an attempt to satisfy these requirements, the value of ψ for the maximum value of ρ being evaluated, was set to one:

$$\psi(\rho_{max}) \;:=\; 1\;. \tag{5.2.19}$$

Graphs of the evaluated conformal factors for two values A, along with the remaining factor from inside the integral in equation (5.2.15), are shown in figure 5.2 below.



(a) $A = 5$ (b) $A = 10$

Figure 5.2: Graph of the conformal factor solution ψ in blue, with the function $\rho\,\Phi(\rho)$ in red, for different values of A, for ρ from $0 \rightarrow 4$.

It can be seen from figure 5.2 above, that the function $\rho\,\Phi(\rho)$ tends exponentially to zero. Within the accuracy of the numerical calculations, this function is essentially zero from $\rho = 4$ onwards.

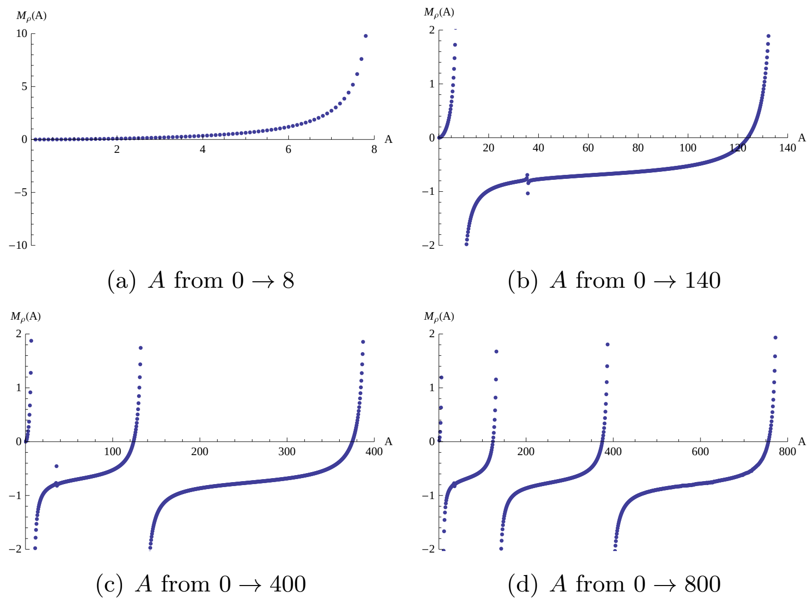
Mass-Factor Approximations with respect to Amplitude A

Mathematica was again used to obtain a graph for the mass-factor M_ρ with respect to the "amplitude" of the Brill metric, given by the constant A. For each value of A, the conformal factor was solved for, and the mass-factor for that value of A was then numerically evaluated, with the conformal factor substituted into equation (5.2.15):

$$M_\rho \;=\; \frac{1}{2}A\int_0^4 \rho\,(2\rho^4 - 5\rho^2 + 1)\; e^{-\rho^2}\;\psi(\rho)\,d\rho\;, \tag{5.2.20}$$

and integrated numerically, with the operation "`NIntegrate`". The integral was evaluated from $\rho = 0$ to $\rho = 4$, since the conformal factor is essentially multiplied by zero for $\rho > 4$, resulting in an integral of "0" between 4 and ∞.

This process was repeated by *Mathematica* for the range of values of A required. A table was formed for all of the computed solutions $M(A)$, and the table plotted. Four of these graphs, for different ranges of A, are shown in figure 5.3 below.



(a) A from $0 \to 8$ (b) A from $0 \to 140$

(c) A from $0 \to 400$ (d) A from $0 \to 800$

Figure 5.3: Graph of $M_\rho(A)$ for different ranges of A.

The final few graphs in particular, show a very close resemblance to figure 5.1, for the Schrödinger scattering analogue. Though the assumptions made, do not satisfy the asymptotic flatness requirement of the space-time, the results for the "mass-factor" M_ρ nevertheless give an indication of the validity of the scattering problem solution.

Chapter 6

Concluding Remarks

6.1 Axially-Symmetric Transverse Trace-Free Tensors

In the $3+1$ formalism for numerical relativity, one of the most common methods of constructing initial data is the conformal transverse trace-free decomposition. This decomposition uses transverse trace-free (TT) tensors for two of the freely specifiable components. The remaining six free components are given by a choice of conformal metric, and the trace of the extrinsic curvature. The Einstein constraint equations are then solved to give the complete set of initial conditions for the formalism.

For a given coordinate system, and a given choice of a conformal spatial metric, a TT tensor can be expressed as a symmetric 3×3 matrix, with the transverse and trace-free conditions restricting the choice of components to two. However, finding a matrix representation for a TT tensor, depending on only two component choices, is not a trivial problem. Hence, in this thesis, the condition of axial symmetry was added to the TT tensors, in order to reduce the complexity of the problem.

Tensors for a Flat Spatial Metric

For a flat spatial metric, expressions were successfully found for TT tensors, depending on two scalar potentials, in both spherical and cylindrical polar coordinates. See equations (3.2.15) and (3.2.27) respectively. Having the tensor depend on two specific functions alone, allows the freedom to make small adjustments to either function, changing the effect of the TT tensor, and thus, the initial conditions resulting from it.

In an attempt to find a coordinate-*free* expression, similar to that given by *Sergio Dain* [19] for a time rotation symmetry, a comparison was made between the scalar potentials of the spherical and cylindrical coordinates. A simple correspondence was found for the potentials which were consistent with the time rotation symmetry used in *Dain* [19]. However, the relations between the remaining potentials involve complicated integrals, making it difficult to find any general expression.

In future studies, it may prove useful to derive an expression in cartesian coordinates. The condition of axial symmetry will not be so easily encompassed, but it may shed light on any covariant structure of the potentials.

Tensors for a General Spatial Metric

For a more general axially symmetric spatial metric, the Brill wave metric from *Brill* [13] was taken, and the technique which was used for the flat space tensors applied. Unfortunately, the condition of axial symmetry did not sufficiently simplify the transverse and trace-free conditions, giving only the scalar potential associated with the time rotation symmetry of *Dain* [19].

The two remaining divergence equations, each contained terms involving three components. With appropriate manipulations of both equations, a combination was found, which removed one of the component dependencies. This resulted in a single, second order partial differential equation, in two of the tensor components. Solving this equation would require two boundary conditions, depending on further information about the system. Numerical approximations may also be necessary, reducing the possibility of finding an specific expression involving a second scalar potential.

Further Investigations on Flat Space Tensors

Despite being the most simple case, the flat spatial metrics prove to give very useful space-times. Since the spatial metric related to the TT tensor is a *conformal* metric, for the conformal transverse trace-free decomposition, the initial hypersurface does not *itself* need to be flat. The commonly used Bowen-York curvature is an example of such a tensor, being divergence-free and trace-free, with respect to a flat 3-metric, which is conformally related to the physical spatial metric.

Calculations here have successfully given the appropriate choice of scalar potentials, for the TT tensor derived in cylindrical coordinates (3.2.27), giving both the linear and angular momentum forms of the Bowen-York curvature, as well as a combination of the two, see equations (3.4.18). The angular momentum case has also been shown to be given by an appropriate choice of the potentials for the tensor in spherical coordinates (3.4.8), which agrees with earlier calculations carried out in *Dain et al.* [21].

A restriction of the TT tensor in cylindrical coordinates (3.2.27), such that the tensor is regular at the origin, has also been derived. The regular TT tensor is given by a new expression (3.5.11), depending on two new scalar potentials, which each have specific restrictions. The regularity ensures that none of the tensor components increase asymptotically, as the origin is approached. This allows components to have non-zero values at the origin, or along the axis of symmetry, without the previous concerns of asymptotic behavior. However, the restrictions on the two scalar functions is quite strong, reducing the usefulness of the expression.

Finally, a condition was given for an axially-symmetric TT tensor to have a spherically symmetric product. This product would allow a reduction of the Hamiltonian constraint to an *ordinary* differential equation, making the conformal factor much easier to solve for. The condition does not easily lead to solution tensors however. Even the additional restriction of time rotation symmetry, requires the solving of a first order, non-linear, partial differential equation for the single remaining potential.

A simplified solution was derived, showing the existence of non-spherically symmetric tensors, with a spherically symmetric product, though the potential for this solution tends to infinity as the radial coordinate increases. The problem could benefit from further work however, with the possibility of further solutions being found, based on adjustments to the given solution.

6.2 Kerr and Bowen-York Extrinsic Curvatures

For numerical relativity, the simplifications due to conformal flatness make the Bowen-York initial data very convenient for the evolution of spinning black hole systems. However, it seems clear that any space-time describing a spinning black hole, will eventually settle down to a Kerr space-time, and that the Kerr black hole does not have a conformally flat slicing. In the evolution of Bowen-York initial data, "spurious" gravitational radiation is therefore emitted, as the black hole tends towards a Kerr space-time, which can interfere with the "physical" radiation being studied.

With many comparisons of the spatial metrics of the Bowen-York and Kerr space-times in the literature, an attempt was made here to compare the *extrinsic curvatures* of the two. This required the calculation of the extrinsic curvature of the decomposed Kerr metric in Boyer-Lindquist coordinates, along with its coordinate derivatives. These calculations are included in the appendix, along with a verification of the expressions obtained, by explicit calculation of the constraint equations.

Since the Bowen-York curvature is a *conformal* curvature, the physical curvature requires the solving of the Hamiltonian constraint for the conformal factor. However, in the limit of asymptotic flatness, the conformal factor can be approximated to 1, with the physical and conformal curvatures equivalent. The difference between the Bowen-York and Kerr extrinsic curvatures was therefore found in this limit, showing the difference to be of order r^{-5} for large values of the radial coordinate r.

Both curvature tensors are transverse and trace-free with respect to their spatial metrics, so the flat divergence of the Kerr curvature should give an indication of how well it could be applied to the Bowen-York spatial metric. Again, difficulties arose due to the conformal factor, but the limit for large r gave the divergence to be of order r^{-7}.

6.3 Brill Wave Mass-Energy

The "spurious" radiation emitted by the settling of a Bowen-York black hole to a Kerr space-time, can be seen to take the form of a Brill wave, see for example *Brandt & Seidel* [12]. The spatial metric describing this wave is given by a conformal metric in *Brill* [13], where it is shown to have positive mass.

The partial differential equation for the conformal factor, known as Lichnerowicz equation, is given in [13] in a form similar to a 3-dimensional Schrödinger equation. This was then expanded upon in *Wheeler* [50], where an analytic solution to a Schrödinger scattering problem was used as an analogue for the Lichnerowicz equation, to give an expression for the *mass* of the Brill wave.

Here, a simplification of both the Brill function $q(\rho, z)$ and the conformal factor was performed, so that Lichnerowicz equation could be reduced to an *ordinary* differential equation. This simplification does not satisfy the requirements for the Brill metric to be asymptotically flat, leading to an infinite measure of mass. However the volume integral for the mass was able to be separated into two parts, with only one of these parts integrating to infinity, and the other seen as a "mass-factor".

Numerical computations carried out in *Mathematica* were able to give a solution for this mass factor, for a range of values of the amplitude A of the wave. The behavior of the graph of this mass factor, with respect to A, was seen to agree with that of the Scrödinger analogue given by Wheeler, giving further evidence of the validity of the Wheeler result.

Further work could be carried out on this problem, by trying to fit the tangent function of Wheeler to the data recorded from the numerical evaluations, to see how closely they agree. A numerical solution to the full *partial* differential equation for the conformal factor, for a select number of values of A, could also give a better indication of the validity of the "mass-factor" calculated here.

Appendix A

Kerr Extrinsic Curvature Calculations

A.1 Extrinsic Curvature for the Natural Kerr Slicing

In this appendix, the Kerr metric, in Boyer-Lindquist coordinates, is fully decomposed into the $3+1$ formalism, according to section 2.1.7, and the extrinsic curvature then calculated directly from equation (2.1.17).

A.1.1 Kerr Space-Time Metric in Boyer-Lindquist Coordinates

The Kerr metric, in Boyer-Lindquist coordinates (t, r, θ, ϕ), is given by equation (1.3.34):

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2Mr}{\Sigma}\right) & 0 & 0 & -\frac{2Mra\ \sin^2\theta}{\Sigma} \\ 0 & \frac{\Sigma}{\triangle} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\frac{2Mra\sin^2\theta}{\Sigma} & 0 & 0 & \left(\frac{(r^2+a^2)^2-\triangle a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta \end{pmatrix} \tag{A.1.1}$$

with $\triangle$ and Σ from (1.3.35):

$$\triangle = r^2 - 2Mr + a^2 \ , \tag{A.1.2a}$$

$$\Sigma = r^2 + a^2\cos^2\theta \ , \tag{A.1.2b}$$

where the angular momentum of the mass J is given by $J = Ma$.

The individual components are given separately by:

$$g_{tt} = -\left(1-\frac{2Mr}{\Sigma}\right) \ , \tag{A.1.3a}$$

$$g_{rr} = \frac{\Sigma}{\triangle} \ , \tag{A.1.3b}$$

$$g_{\theta\theta} = \Sigma \ , \tag{A.1.3c}$$

$$g_{\phi\phi} = \left(\frac{(r^2+a^2)^2-\triangle a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta \ , \tag{A.1.3d}$$

$$g_{t\phi} = g_{\phi t} = -\frac{2Mra\sin^2\theta}{\Sigma} \ , \tag{A.1.3e}$$

for easy distinguishing and referencing.

A.1.2 The Inverse of the Kerr Metric

For a metric in the form of the Kerr metric (A.1.1):

$$\begin{pmatrix} g_{00} & 0 & 0 & g_{03} \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ g_{03} & 0 & 0 & g_{33} \end{pmatrix}, \tag{A.1.4}$$

the determinant g is given by the component expression:

$$g \;=\; g_{00}g_{11}g_{22}g_{33} - {g_{03}}^2 g_{11}g_{22} \;, \tag{A.1.5}$$

and the inverse $g^{\mu\nu}$, by:

$$g^{\mu\nu} \;=\; \frac{1}{g}\begin{pmatrix} g_{11}g_{22}g_{33} & 0 & 0 & -g_{03}g_{11}g_{22} \\ 0 & \begin{array}{c} -{g_{03}}^2 g_{22} \\ +g_{00}g_{22}g_{33} \end{array} & 0 & 0 \\ 0 & 0 & \begin{array}{c} -{g_{03}}^2 g_{11} \\ +g_{00}g_{11}g_{33} \end{array} & 0 \\ -g_{03}g_{11}g_{22} & 0 & 0 & g_{00}g_{11}g_{22} \end{pmatrix}. \tag{A.1.6}$$

The individual components, for the Boyer-Lindquist coordinates (t, r, θ, ϕ), are then given by:

$$g^{rr} \;=\; \frac{\cancel{g_{00}g_{22}g_{33} - {g_{03}}^2 g_{22}}}{g_{11}\cancel{\left(g_{00}g_{22}g_{33} - {g_{03}}^2 g_{22}\right)}} \;=\; \frac{1}{g_{rr}} \;, \tag{A.1.7a}$$

$$g^{\theta\theta} \;=\; \frac{\cancel{g_{00}g_{11}g_{33} - {g_{03}}^2 g_{11}}}{g_{22}\cancel{\left(g_{00}g_{11}g_{33} - {g_{03}}^2 g_{11}\right)}} \;=\; \frac{1}{g_{\theta\theta}} \;, \tag{A.1.7b}$$

$$g^{tt} \;=\; \frac{\cancel{g_{11}g_{22}}g_{33}}{g_{00}\cancel{g_{11}g_{22}}g_{33} - {g_{03}}^2\cancel{g_{11}g_{22}}} \;=\; \frac{g_{\phi\phi}}{g_{tt}g_{\phi\phi} - {g_{t\phi}}^2} \;, \tag{A.1.7c}$$

$$g^{t\phi} \;=\; g^{\phi t} \;=\; \frac{-g_{03}\cancel{g_{11}g_{22}}}{g_{00}\cancel{g_{11}g_{22}}g_{33} - {g_{03}}^2\cancel{g_{11}g_{22}}} \;=\; \frac{-g_{t\phi}}{g_{tt}g_{\phi\phi} - {g_{t\phi}}^2} \;, \tag{A.1.7d}$$

$$g^{\phi\phi} \;=\; \frac{g_{00}\cancel{g_{11}g_{22}}}{g_{00}\cancel{g_{11}g_{22}}g_{33} - {g_{03}}^2\cancel{g_{11}g_{22}}} \;=\; \frac{g_{tt}}{g_{tt}g_{\phi\phi} - {g_{t\phi}}^2}, \tag{A.1.7e}$$

showing the components g^{rr} and $g^{\theta\theta}$ to be given simply by the inverses of the components g_{rr} and $g_{\theta\theta}$ respectively. The remaining three components require a bit more calculation however.

Evaluating the inverse metric components g^{tt}, $g^{t\phi}$ and $g^{\phi\phi}$ explicitly, by substituting from equations (A.1.3):

$$\begin{aligned}
g^{tt} &= \frac{g_{\phi\phi}}{g_{tt}g_{\phi\phi} - {g_{t\phi}}^2} \\
&= \frac{\left(\frac{(r^2+a^2)^2-(r^2-2Mr+a^2)a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta}{-\left(1-\frac{2Mr}{\Sigma}\right)\left(\frac{(r^2+a^2)^2-(r^2-2Mr+a^2)a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta - \frac{4a^2M^2r^2\sin^4\theta}{\Sigma^2}} \\
&= \frac{\left((r^2+a^2)^2-(r^2-2Mr+a^2)a^2\sin^2\theta\right)\Sigma}{-\left(\Sigma-2Mr\right)\left((r^2+a^2)^2-(r^2-2Mr+a^2)a^2\sin^2\theta\right) - 4a^2M^2r^2\sin^2\theta}\ ,
\end{aligned} \tag{A.1.8a}$$

$$\begin{aligned}
g^{t\phi} &= \frac{-g_{t\phi}}{g_{tt}g_{\phi\phi} - {g_{t\phi}}^2} \\
&= \frac{\frac{2rMa\sin^2\theta}{\Sigma}}{-\left(1-\frac{2Mr}{\Sigma}\right)\left(\frac{(r^2+a^2)^2-(r^2-2Mr+a^2)a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta - \frac{4a^2M^2r^2\sin^4\theta}{\Sigma^2}} \\
&= \frac{2rMa\Sigma}{-\left(\Sigma-2Mr\right)\left((r^2+a^2)^2-(r^2-2Mr+a^2)a^2\sin^2\theta\right) - 4a^2M^2r^2\sin^2\theta}\ ,
\end{aligned} \tag{A.1.8b}$$

$$\begin{aligned}
g^{\phi\phi} &= \frac{g_{tt}}{g_{tt}g_{\phi\phi} - {g_{t\phi}}^2} \\
&= \frac{-\left(1-\frac{2Mr}{\Sigma}\right)}{-\left(1-\frac{2Mr}{\Sigma}\right)\left(\frac{(r^2+a^2)^2-(r^2-2Mr+a^2)a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta - \frac{4a^2M^2r^2\sin^4\theta}{\Sigma^2}} \\
&= \frac{-\frac{\Sigma(\Sigma-2Mr)}{\sin^2\theta}}{-\left(\Sigma-2Mr\right)\left((r^2+a^2)^2-(r^2-2Mr+a^2)a^2\sin^2\theta\right) - 4a^2M^2r^2\sin^2\theta}\ .
\end{aligned} \tag{A.1.8c}$$

Since the denominators for each of equations (A.1.8) are equivalent, the expression can be simplified separately:

$$
\begin{aligned}
&-(\Sigma - 2Mr)\left((r^2+a^2)^2 - (r^2-2Mr+a^2)a^2\sin^2\theta\right) - 4a^2M^2r^2\sin^2\theta \\
=&\ -(\Sigma - 2Mr)\left(r^4 + 2r^2a^2 + a^4 - r^2a^2\sin^2\theta + 2rMa^2\sin^2\theta - a^4\sin^2\theta\right) \\
&\ -4a^2M^2r^2\sin^2\theta \\
=&\ -(\Sigma - 2Mr)\left(r^4 + r^2a^2(2-\sin^2\theta) + a^4(1-\sin^2\theta) + 2rMa^2\sin^2\theta\right) \\
&\ -4r^2M^2a^2\sin^2\theta \\
=&\ -\Sigma\left(r^4 + r^2a^2(1+\cos^2\theta) + a^4\cos^2\theta + 2rMa^2\sin^2\theta\right) \\
&\ +2r^5M + 2r^3Ma^2(1+\cos^2\theta) + 2rMa^4\cos^2\theta \\
&\ +\cancel{4r^2M^2a^2\sin^2\theta} - \cancel{4r^2M^2a^2\sin^2\theta} \qquad\qquad \text{(A.1.9)} \\
=&\ -\Sigma\left(r^4 + r^2a^2(1+\cos^2\theta) + a^4\cos^2\theta + 2rMa^2\sin^2\theta\right) \\
&\ +2rM(r^2+a^2)(r^2+a^2\cos^2\theta) \\
=&\ -\Sigma\left(r^4 - 2r^3M + r^2a^2 + r^2a^2\cos^2\theta - 2rMa^2\cos^2\theta + a^4\cos^2\theta\right) \\
=&\ -\Sigma^2(r^2 - 2rM + a^2) \\
=&\ -\Sigma^2\triangle\ .
\end{aligned}
$$

The complete expressions for the inverse metric components can now be given, by substituting from equations (A.1.3) into (A.1.7) for g^{rr} and $g^{\theta\theta}$, and by substituting the simplified expression for the denominator (A.1.9) into equations (A.1.8):

$$g^{rr} = \frac{1}{g_{rr}} = \frac{\triangle}{\Sigma} \, , \tag{A.1.10a}$$

$$g^{\theta\theta} = \frac{1}{g_{\theta\theta}} = \frac{1}{\Sigma} \, , \tag{A.1.10b}$$

$$\begin{aligned} g^{tt} &= \frac{\left((r^2+a^2)^2 - (r^2 - 2Mr + a^2)a^2 \sin^2\theta\right)\Sigma}{-\Sigma^2 \triangle} \\ &= -\frac{\left((r^2+a^2)^2 - (r^2 - 2Mr + a^2)a^2 \sin^2\theta\right)}{\Sigma \triangle} \, , \end{aligned} \tag{A.1.10c}$$

$$g^{t\phi} = \frac{2rMa\Sigma}{-\Sigma^2\triangle} = -\frac{2rMa}{\Sigma\triangle} \, , \tag{A.1.10d}$$

$$g^{\phi\phi} = \frac{-\frac{\Sigma(\Sigma-2Mr)}{\sin^2\theta}}{-\Sigma^2\triangle} = \frac{\Sigma - 2Mr}{\Sigma\triangle\sin^2\theta} \, . \tag{A.1.10e}$$

In matrix form, the inverse of the Kerr metric (A.1.1), in Boyer-Lindquist coordinates (t, r, θ, ϕ), is given by:

$$g^{\mu\nu} = \begin{pmatrix} -\frac{\left((r^2+a^2)^2 - \triangle a^2 \sin^2\theta\right)}{\Sigma\triangle} & 0 & 0 & -\frac{2rMa}{\Sigma\triangle} \\ 0 & \frac{\triangle}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ -\frac{2rMa}{\Sigma\triangle} & 0 & 0 & \frac{\Sigma-2Mr}{\Sigma\triangle\sin^2\theta} \end{pmatrix} , \tag{A.1.11}$$

with $\triangle$ and Σ still given from equation (A.1.2) by:

$$\triangle = r^2 - 2Mr + a^2 \, , \tag{A.1.12a}$$

$$\Sigma = r^2 + a^2\cos^2\theta \, . \tag{A.1.12b}$$

A.1.3 Coordinate Derivatives of the Metric Components

The derivatives of the components (A.1.3) of the Kerr metric, with respect to r, are given by:

$$\begin{aligned}
\frac{\partial g_{rr}}{\partial x^r} &= \frac{\partial}{\partial x^r}\left(\frac{r^2 + a^2\cos^2\theta}{r^2 - 2rM + a^2}\right) \\
&= \frac{(r^2 - 2rM + a^2)(2r) - (r^2 + a^2\cos^2\theta)(2r - 2M)}{(r^2 - 2rM + a^2)^2} \\
&= \frac{2(-r^2M + ra^2\sin^2\theta + Ma^2\cos^2\theta)}{(r^2 - 2rM + a^2)^2}\ ,
\end{aligned} \tag{A.1.13a}$$

$$\begin{aligned}
\frac{\partial g_{\theta\theta}}{\partial x^r} &= \frac{\partial}{\partial x^r}\left(r^2 + a^2\cos^2\theta\right) \\
&= 2r\ ,
\end{aligned} \tag{A.1.13b}$$

$$\begin{aligned}
\frac{\partial g_{\phi\phi}}{\partial x^r} &= \frac{\partial}{\partial r}\left(\frac{(r^2 + a^2)^2 - (r^2 - 2rM + a^2)a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\sin^2\theta \\
&= \frac{\sin^2\theta}{\Sigma^2}\Big((r^2 + a^2\cos^2\theta)\left(4r(r^2 + a^2) - (2r - 2M)a^2\sin^2\theta\right) \\
&\quad - 2r\left((r^2 + a^2)^2 - (r^2 - 2rM + a^2)a^2\sin^2\theta\right)\Big) \\
&= \frac{2\sin^2\theta\left(r^5 + 2r^3a^2\cos^2\theta - r^2Ma^2\sin^2\theta + ra^4\cos^4\theta + Ma^4\cos^2\theta\ \sin^2\theta\right)}{\Sigma^2} \\
&= \frac{2\sin^2\theta\left(r\Sigma^2 - Ma^2\sin^2\theta(r^2 - a^2\cos^2\theta)\right)}{\Sigma^2}\ ,
\end{aligned} \tag{A.1.13c}$$

$$\begin{aligned}
\frac{\partial g_{\phi t}}{\partial x^r} &= \frac{\partial}{\partial r}\left(-\frac{2rMa\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) \\
&= -\frac{(r^2 + a^2\cos^2\theta)\ (2Ma\sin^2\theta) - (2rMa\sin^2\theta)\ 2r}{(r^2 + a^2\cos^2\theta)^2} \\
&= \frac{2Ma\sin^2\theta\ (r^2 - a^2\cos^2\theta)}{\Sigma^2}\ .
\end{aligned} \tag{A.1.13d}$$

The derivatives of the components (A.1.3), with respect to θ, are given by:

$$\frac{\partial g_{rr}}{\partial x^\theta} = \frac{\partial}{\partial x^\theta}\left(\frac{r^2 + a^2\cos^2\theta}{(r^2 - 2rM + a^2)}\right)$$

$$= \frac{-2a^2\cos\theta\ \sin\theta}{(r^2 - 2rM + a^2)}\ , \tag{A.1.14a}$$

$$\frac{\partial g_{\theta\theta}}{\partial x^\theta} = \frac{\partial}{\partial x^\theta}\left(r^2 + a^2\cos^2\theta\right)$$

$$= -\,2a^2\cos\theta\ \sin\theta\ , \tag{A.1.14b}$$

$$\begin{aligned}\frac{\partial g_{\phi\phi}}{\partial x^\theta} &= \frac{\partial}{\partial\theta}\left(\frac{(r^2+a^2)^2 - (r^2 - 2rM + a^2)a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\sin^2\theta \\ &= \frac{1}{\Sigma^2}\Bigg((r^2 + a^2\cos^2\theta) \\ &\qquad \left(2(r^2+a^2)^2\sin\theta\ \cos\theta - 4(r^2 - 2rM + a^2)a^2\sin^3\theta\ \cos\theta\right) \\ &\quad + \left((r^2+a^2)^2\sin^2\theta - (r^2 - 2rM + a^2)a^2\sin^4\theta\right)\ 2a^2\cos\theta\ \sin\theta\Bigg) \\ &= \frac{2\sin\theta\ \cos\theta}{\Sigma^2}\Bigg((r^2 + a^2\cos^2\theta)\left((r^2+a^2)^2 - 2(r^2 - 2rM + a^2)a^2\sin^2\theta\right) \\ &\quad + a^2\sin^2\theta\left((r^2+a^2)^2 - (r^2 - 2rM + a^2)a^2\sin^2\theta\right)\Bigg) \\ &= \frac{2\sin\theta\ \cos\theta}{\Sigma^2}\left(\left(r^2+a^2\right)^3 - (\Sigma + r^2 + a^2)(r^2 - 2rM + a^2)a^2\sin^2\theta\right)\ ,\end{aligned} \tag{A.1.14c}$$

$$\begin{aligned}\frac{\partial g_{\phi t}}{\partial x^\theta} &= \frac{\partial}{\partial\theta}\left(-\frac{2rMa\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) \\ &= -\ \frac{(r^2 + a^2\cos^2\theta)\ (4rMa\sin\theta\ \cos\theta) + (2rMa\sin^2\theta)\ (2a^2\cos\theta\ \sin\theta)}{(r^2 + a^2\cos^2\theta)^2}\end{aligned}$$

$$= -\ \frac{4rMa\sin\theta\ \cos\theta\ (r^2 + a^2)}{\Sigma^2}\ . \tag{A.1.14d}$$

A.1.4 Lapse, Shift and Spatial Metric

As outlined in section 2.1.7, the lapse and shift, and the spatial metric of the Kerr metric (A.1.1), in Boyer-Lindquist coordinates, are found by equating the metric to equation (2.1.14):

$$\begin{pmatrix} g_{tt} & g_{tj} \\ g_{it} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_s \beta^s & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix} ,$$

$$\begin{pmatrix} g^{tt} & g^{tj} \\ g^{it} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix} . \tag{A.1.15}$$

The lapse can therefore be given, in terms of the metric components, by:

$$\alpha = \sqrt{-\frac{g_{tt} g_{\phi\phi} - {g_{t\phi}}^2}{g_{\phi\phi}}} , \tag{A.1.16}$$

and from this, the shift given by:

$$\begin{aligned} \beta^i &= \alpha^2 \left(0, 0, \frac{-g_{t\phi}}{g_{tt} g_{\phi\phi} - {g_{t\phi}}^2} \right) \\ &= \left(0, 0, -\frac{g_{tt} g_{\phi\phi} - {g_{t\phi}}^2}{g_{\phi\phi}} \frac{-g_{t\phi}}{g_{tt} g_{\phi\phi} - {g_{t\phi}}^2} \right) \\ &= \left(0, 0, \frac{g_{t\phi}}{g_{\phi\phi}} \right) . \end{aligned} \tag{A.1.17}$$

The unit time-like normal, and its covector, are then given by:

$$\begin{aligned} n^\mu &= \left(\frac{1}{\alpha}, 0, 0, \frac{-\beta^\phi}{\alpha} \right) \\ &= \frac{1}{\alpha} \left(1, 0, 0, -\beta^\phi \right) \\ &= \sqrt{-\frac{g_{\phi\phi}}{g_{tt} g_{\phi\phi} - {g_{t\phi}}^2}} \left(1, 0, 0, \frac{g_{t\phi}}{g_{\phi\phi}} \right) , \end{aligned} \tag{A.1.18a}$$

$$\begin{aligned} n_\mu &= (-\alpha, 0, 0, 0) \\ &= -\sqrt{-\frac{g_{tt} g_{\phi\phi} - {g_{t\phi}}^2}{g_{\phi\phi}}} (1, 0, 0, 0) , \end{aligned} \tag{A.1.18b}$$

again in terms of the metric components, which can be evaluated explicitly by substituting from equations (A.1.3).

A.1.5 Evaluation of the Extrinsic Curvature

The extrinsic curvature K_{ab}, can be seen from equation (2.1.17), to be given by the Lie derivative of the spatial metric γ_{ab}, with respect to the unit time-like normal n^{μ}:

$$K_{ab} = \frac{1}{2}\,\mathcal{L}_n \gamma_{ab} \ . \tag{A.1.19}$$

Working with the 4 space-time indices, for evaluation of the Lie derivative with respect to the *time-like* vector n^{μ}, the space-time definition (2.1.7) of the spatial metric:

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_{\mu}\, n_{\nu} \ , \tag{A.1.20}$$

is used to evaluate the Lie derivative, by equations (1.2.52), (1.2.48) and the fact that the Lie derivative is a linear operator:

$$\begin{aligned} K_{\mu\nu} &= \frac{1}{2}\,\mathcal{L}_n\ (g_{\mu\nu} + n_{\mu} n_{\nu}) \\ &= \frac{1}{2}\,\mathcal{L}_n g_{\mu\nu} + \frac{1}{2}\ (\mathcal{L}_n n_{\mu})\, n_{\nu} + \frac{1}{2}\, n_{\mu}\,(\mathcal{L}_n n_{\nu}) \\ &= \frac{1}{2}\Big(\nabla_{\mu} n_{\nu} + \nabla_{\nu} n_{\mu} \\ &\quad + n_{\nu}\left(n^{\lambda}\nabla_{\lambda} n_{\nu} + n_{\lambda}\nabla_{\nu} n^{\lambda}\right) \\ &\quad + n_{\mu}\left(n^{\lambda}\nabla_{\lambda} n_{\mu} + n_{\lambda}\nabla_{\mu} n^{\lambda}\right)\Big) \ , \end{aligned} \tag{A.1.21}$$

noting that the covariant derivative is taken to be associated with the metric $g_{\mu\nu}$, hence its covariant derivative is zero.

The covariant derivatives of the time-like normal vector n^{μ} and its covector n_{μ} are given, from equations (1.2.32), (1.2.33) for the derivatives and (1.2.36) for the connection coefficients:

$$\begin{aligned} \nabla_{\beta} n_{\alpha} &= \partial_{\beta} n_{\alpha} \ - \ \Gamma^{\kappa}_{\beta\alpha} n_{\kappa} \\ &= \partial_{\beta} n_{\alpha} \ - \ \frac{1}{2} g^{\kappa\eta}\left(\frac{\partial g_{\beta\eta}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\eta}}{\partial x^{\beta}} - \frac{\partial g_{\beta\alpha}}{\partial x^{\eta}}\right) n_{\kappa} \ , \end{aligned} \tag{A.1.22}$$

$$\begin{aligned} \nabla_{\beta} n^{\alpha} &= \partial_{\beta} n^{\alpha} \ + \ \Gamma^{\alpha}_{\beta\kappa} n^{\kappa} \\ &= \partial_{\beta} n^{\alpha} \ + \ \frac{1}{2} g^{\alpha\eta}\left(\frac{\partial g_{\beta\eta}}{\partial x^{\kappa}} + \frac{\partial g_{\kappa\eta}}{\partial x^{\beta}} - \frac{\partial g_{\beta\kappa}}{\partial x^{\eta}}\right) n^{\kappa} \ , \end{aligned} \tag{A.1.23}$$

which can now be used for the further evaluation of equation (A.1.21).

Substituting (A.1.23) and (A.1.22) into equation (A.1.21), and expanding gradually:

$$
\begin{aligned}
K_{\mu\nu} &= \frac{1}{2}\Bigg(\left(\partial_\mu n_\nu \; - \; \Gamma^\kappa{}_{\mu\nu} n_\kappa\right) + \left(\partial_\nu n_\mu \; - \; \Gamma^\kappa{}_{\nu\mu} n_\kappa\right) \\
&\quad + n_\nu n^\lambda \left(\partial_\lambda n_\nu \; - \; \Gamma^\kappa{}_{\lambda\nu} n_\kappa\right) + n_\nu n_\lambda \left(\partial_\nu n^\lambda \; + \; \Gamma^\lambda{}_{\nu\kappa} n^\kappa\right) \\
&\quad + n_\mu n^\lambda \left(\partial_\lambda n_\mu \; - \; \Gamma^\kappa{}_{\lambda\mu} n_\kappa\right) + n_\mu n_\lambda \left(\partial_\mu n^\lambda \; + \; \Gamma^\lambda{}_{\mu\kappa} n^\kappa\right) \Bigg) \\
&= \frac{1}{2}\Bigg(\partial_\mu n_\nu - \Gamma^\kappa{}_{\mu\nu} n_\kappa + \partial_\nu n_\mu - \Gamma^\kappa{}_{\nu\mu} n_\kappa \\
&\quad + n_\nu n^\lambda \partial_\lambda n_\nu - \Gamma^\kappa{}_{\lambda\nu} \; n_\nu n^\lambda n_\kappa + n_\nu n_\lambda \partial_\nu n^\lambda + \Gamma^\lambda{}_{\nu\kappa} \; n_\nu n_\lambda n^\kappa \\
&\quad + n_\mu n^\lambda \partial_\lambda n_\mu - \Gamma^\kappa{}_{\lambda\mu} \; n_\mu n^\lambda n_\kappa + n_\mu n_\lambda \partial_\mu n^\lambda + \Gamma^\lambda{}_{\mu\kappa} \; n_\mu n_\lambda n^\kappa \Bigg) \\
&= \frac{1}{2}\Bigg(\partial_\mu n_\nu - \frac{1}{2} g^{\kappa\eta}\left(\frac{\partial g_{\mu\eta}}{\partial x^\nu} + \frac{\partial g_{\nu\eta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\eta}\right) n_\kappa \\
&\quad + \partial_\nu n_\mu - \frac{1}{2} g^{\kappa\eta}\left(\frac{\partial g_{\nu\eta}}{\partial x^\mu} + \frac{\partial g_{\mu\eta}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\eta}\right) n_\kappa \\
&\quad + n_\nu n^\lambda \partial_\lambda n_\nu - \frac{1}{2} g^{\kappa\eta}\left(\frac{\partial g_{\lambda\eta}}{\partial x^\nu} + \frac{\partial g_{\nu\eta}}{\partial x^\lambda} - \frac{\partial g_{\lambda\nu}}{\partial x^\eta}\right) \; n_\nu n^\lambda n_\kappa \\
&\quad + n_\nu n_\lambda \partial_\nu n^\lambda + \frac{1}{2} g^{\lambda\eta}\left(\frac{\partial g_{\nu\eta}}{\partial x^\kappa} + \frac{\partial g_{\kappa\eta}}{\partial x^\nu} - \frac{\partial g_{\nu\kappa}}{\partial x^\eta}\right) \; n_\nu n_\lambda n^\kappa \\
&\quad + n_\mu n^\lambda \partial_\lambda n_\mu - \frac{1}{2} g^{\kappa\eta}\left(\frac{\partial g_{\lambda\eta}}{\partial x^\mu} + \frac{\partial g_{\mu\eta}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\eta}\right) \; n_\mu n^\lambda n_\kappa \\
&\quad + n_\mu n_\lambda \partial_\mu n^\lambda + \frac{1}{2} g^{\lambda\eta}\left(\frac{\partial g_{\mu\eta}}{\partial x^\kappa} + \frac{\partial g_{\kappa\eta}}{\partial x^\mu} - \frac{\partial g_{\mu\kappa}}{\partial x^\eta}\right) \; n_\mu n_\lambda n^\kappa \Bigg) \, .
\end{aligned}
\tag{A.1.24}
$$

Equation (A.1.24) is now summed over the three indices which only appear on the left of the equation, λ, κ and η:

Summing over $\lambda \in \{t, r, \theta, \phi\}$:

$$
\begin{aligned}
K_{\mu\nu} \;&= \frac{1}{2}\Bigg(\partial_\mu n_\nu - \frac{1}{2}\; n_\kappa \; g^{\kappa\eta} \left(\frac{\partial g_{\mu\eta}}{\partial x^\nu} + \frac{\partial g_{\nu\eta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\eta} \right) \\
&\qquad + \partial_\nu n_\mu - \frac{1}{2}\; n_\kappa \; g^{\kappa\eta} \left(\frac{\partial g_{\nu\eta}}{\partial x^\mu} + \frac{\partial g_{\mu\eta}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\eta} \right) \\
&\qquad + n_\nu n^t n_{\nu,t} - \frac{1}{2}\; n_\nu n^t n_\kappa \; g^{\kappa\eta} \left(\frac{\partial g_{t\eta}}{\partial x^\nu} + \frac{\partial g_{\nu\eta}}{\partial x^t} - \frac{\partial g_{t\nu}}{\partial x^\eta} \right) \\
&\qquad + n_\nu n^\phi n_{\nu,\phi} - \frac{1}{2}\; n_\nu n^\phi n_\kappa \; g^{\kappa\eta} \left(\frac{\partial g_{\phi\eta}}{\partial x^\nu} + \frac{\partial g_{\nu\eta}}{\partial x^\phi} - \frac{\partial g_{\phi\nu}}{\partial x^\eta} \right) \\
&\qquad + n_\nu n_t n^t{}_{,\nu} + \frac{1}{2}\; n_\nu n_t n^\kappa \; g^{t\eta} \left(\frac{\partial g_{\nu\eta}}{\partial x^\kappa} + \frac{\partial g_{\kappa\eta}}{\partial x^\nu} - \frac{\partial g_{\nu\kappa}}{\partial x^\eta} \right) \\
&\qquad + n_\mu n^t n_{\mu,t} - \frac{1}{2}\; n_\mu n^t n_\kappa \; g^{\kappa\eta} \left(\frac{\partial g_{t\eta}}{\partial x^\mu} + \frac{\partial g_{\mu\eta}}{\partial x^t} - \frac{\partial g_{t\mu}}{\partial x^\eta} \right) \\
&\qquad + n_\mu n^\phi n_{\mu,\phi} - \frac{1}{2}\; n_\mu n^\phi n_\kappa \; g^{\kappa\eta} \left(\frac{\partial g_{\phi\eta}}{\partial x^\mu} + \frac{\partial g_{\mu\eta}}{\partial x^\phi} - \frac{\partial g_{\phi\mu}}{\partial x^\eta} \right) \\
&\qquad + n_\mu n_t n^t{}_{,\mu} + \frac{1}{2}\; n_\mu n_t n^\kappa \; g^{t\eta} \left(\frac{\partial g_{\mu\eta}}{\partial x^\kappa} + \frac{\partial g_{\kappa\eta}}{\partial x^\mu} - \frac{\partial g_{\mu\kappa}}{\partial x^\eta} \right) \Bigg) .
\end{aligned}
\tag{A.1.25}
$$

Summing (A.1.25) over $\kappa \in \{t, r, \theta, \phi\}$:

$$
\begin{aligned}
K_{\mu\nu} \;&= \frac{1}{2}\Bigg(\partial_\mu n_\nu + \partial_\nu n_\mu \\
&\quad + n_\nu n^t n_{\nu,t} + n_\nu n^\phi n_{\nu,\phi} + n_\nu n_t n^t{}_{,\nu} + n_\mu n^t n_{\mu,t} + n_\mu n^\phi n_{\mu,\phi} + n_\mu n_t n^t{}_{,\mu} \\
&\quad - \; n_t \; g^{t\eta} \left(\frac{\partial g_{\mu\eta}}{\partial x^\nu} + \frac{\partial g_{\nu\eta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\eta} \right) \\
&\cancel{- \frac{1}{2}\; n_\nu n^t n_t \; g^{t\eta} \left(\frac{\partial g_{t\eta}}{\partial x^\nu} + \frac{\partial g_{\nu\eta}}{\partial x^t} - \frac{\partial g_{t\nu}}{\partial x^\eta} \right)} \cancel{- \frac{1}{2}\; n_\nu n^\phi n_t \; g^{t\eta} \left(\frac{\partial g_{\phi\eta}}{\partial x^\nu} + \frac{\partial g_{\nu\eta}}{\partial x^\phi} - \frac{\partial g_{\phi\nu}}{\partial x^\eta} \right)} \\
&\cancel{+ \frac{1}{2}\; n_\nu n_t n^t \; g^{t\eta} \left(\frac{\partial g_{\nu\eta}}{\partial x^t} + \frac{\partial g_{t\eta}}{\partial x^\nu} - \frac{\partial g_{\nu t}}{\partial x^\eta} \right)} \cancel{+ \frac{1}{2}\; n_\nu n_t n^\phi \; g^{t\eta} \left(\frac{\partial g_{\nu\eta}}{\partial x^\phi} + \frac{\partial g_{\phi\eta}}{\partial x^\nu} - \frac{\partial g_{\nu\phi}}{\partial x^\eta} \right)} \\
&\cancel{- \frac{1}{2}\; n_\mu n^t n_t \; g^{t\eta} \left(\frac{\partial g_{t\eta}}{\partial x^\mu} + \frac{\partial g_{\mu\eta}}{\partial x^t} - \frac{\partial g_{t\mu}}{\partial x^\eta} \right)} \cancel{- \frac{1}{2}\; n_\mu n^\phi n_t \; g^{t\eta} \left(\frac{\partial g_{\phi\eta}}{\partial x^\mu} + \frac{\partial g_{\mu\eta}}{\partial x^\phi} - \frac{\partial g_{\phi\mu}}{\partial x^\eta} \right)} \\
&\cancel{+ \frac{1}{2}\; n_\mu n_t n^t \; g^{t\eta} \left(\frac{\partial g_{\mu\eta}}{\partial x^t} + \frac{\partial g_{t\eta}}{\partial x^\mu} - \frac{\partial g_{\mu t}}{\partial x^\eta} \right)} \cancel{+ \frac{1}{2}\; n_\mu n_t n^\phi \; g^{t\eta} \left(\frac{\partial g_{\mu\eta}}{\partial x^\phi} + \frac{\partial g_{\phi\eta}}{\partial x^\mu} - \frac{\partial g_{\mu\phi}}{\partial x^\eta} \right)} \Bigg).
\end{aligned}
\tag{A.1.26}
$$

Summing (A.1.26) over $\eta \in \{t, r, \theta, \phi\}$:

$$\begin{aligned} K_{\mu\nu} \; = \; & \frac{1}{2}\Big(\partial_\mu n_\nu + \partial_\nu n_\mu \\ & + n_\nu n^t n_{\nu,t} + n_\nu n^\phi n_{\nu,\phi} + n_\nu n_t n^t{}_{,\nu} + n_\mu n^t n_{\mu,t} + n_\mu n^\phi n_{\mu,\phi} + n_\mu n_t n^t{}_{,\mu} \\ & - \; n_t \; g^{tt}\left(\frac{\partial g_{\mu t}}{\partial x^\nu} + \frac{\partial g_{\nu t}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^t}\right) - \; n_t \; g^{t\phi}\left(\frac{\partial g_{\mu\phi}}{\partial x^\nu} + \frac{\partial g_{\nu\phi}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\phi}\right)\Big) \, . \end{aligned} \tag{A.1.27}$$

Since the only non-zero term for n_μ is $\mu = t$, and substituting in for $n_t = -\alpha$, equation (A.1.27), reduces to:

$$K_{\mu\nu} \; = \frac{1}{2}\alpha\Big(g^{tt}\left(\frac{\partial g_{\mu t}}{\partial x^\nu} + \frac{\partial g_{\nu t}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^t}\right) + g^{t\phi}\left(\frac{\partial g_{\mu\phi}}{\partial x^\nu} + \frac{\partial g_{\nu\phi}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\phi}\right)\Big) \, , \tag{A.1.28}$$

giving a final expression for the extrinsic curvature of space-like hypersurfaces in the $3+1$ formalism, for a metric of the form (A.1.1). Evaluating the individual *spatial* components of equation (A.1.28):

$$\begin{aligned} K_{rr} \; & = \frac{1}{2}\alpha\Big(g^{tt}\Big(- \cancelto{0}{\frac{\partial g_{rr}}{\partial t}}\Big) + g^{t\phi}\Big(- \cancelto{0}{\frac{\partial g_{rr}}{\partial \phi}}\Big)\Big) \\ & = 0 \, , \end{aligned} \tag{A.1.29a}$$

$$\begin{aligned} K_{\theta\theta} \; & = \frac{1}{2}\alpha\Big(g^{tt}\Big(- \cancelto{0}{\frac{\partial g_{\theta\theta}}{\partial t}}\Big) + g^{t\phi}\Big(- \cancelto{0}{\frac{\partial g_{\theta\theta}}{\partial \phi}}\Big)\Big) \\ & = 0 \, , \end{aligned} \tag{A.1.29b}$$

$$\begin{aligned} K_{r\theta} \; & = \frac{1}{2}\alpha\Big(g^{tt}\,(0) + g^{t\phi}\,(0)\Big) \\ & = 0 \, , \end{aligned} \tag{A.1.29c}$$

(A.1.29d)

$$\begin{aligned} K_{\phi\phi} \; & = \frac{1}{2}\alpha\Big(g^{tt}\Big(2\cancelto{0}{\frac{\partial g_{\phi t}}{\partial \phi}} - \cancelto{0}{\frac{\partial g_{\phi\phi}}{\partial t}}\Big) + g^{t\phi}\Big(2\cancelto{0}{\frac{\partial g_{\phi\phi}}{\partial \phi}} - \cancelto{0}{\frac{\partial g_{\phi\phi}}{\partial \phi}}\Big)\Big) \\ & = 0 \, , \end{aligned} \tag{A.1.29e}$$

$$K_{r\phi} \; = \frac{1}{2}\alpha\Big(g^{tt}\left(\frac{\partial g_{\phi t}}{\partial r}\right) + g^{t\phi}\left(\frac{\partial g_{\phi\phi}}{\partial r}\right)\Big) \, , \tag{A.1.29f}$$

$$K_{\theta\phi} \; = \frac{1}{2}\alpha\Big(g^{tt}\left(\frac{\partial g_{\phi t}}{\partial \theta}\right) + g^{t\phi}\left(\frac{\partial g_{\phi\phi}}{\partial \theta}\right)\Big) \, , \tag{A.1.29g}$$

giving the components of the extrinsic curvature for the Kerr metric.

A.1.6 Calculation of Individual Components

The non-zero components of the curvature from (A.1.29) are now evaluated explicitly for the Kerr metric (A.1.1). Beginning with (A.1.29f) and substituting for the inverse metric components (A.1.10), and the metric derivatives with respect to r, (A.1.13):

$$
\begin{aligned}
K_{r\phi} &= \frac{1}{2}\alpha\bigg(g^{tt}\left(\frac{\partial g_{\phi t}}{\partial r}\right) + g^{t\phi}\left(\frac{\partial g_{\phi\phi}}{\partial r}\right)\bigg) \\
&= \frac{1}{2}\alpha\bigg(-\frac{((r^2+a^2)^2-(r^2-2Mr+a^2)a^2\sin^2\theta)}{\Sigma\triangle}\left(\frac{2Ma\sin^2\theta\;(r^2-a^2\cos^2\theta)}{\Sigma^2}\right) \\
&\quad -\frac{2rMa}{\Sigma\triangle}\left(\frac{2\sin^2\theta\,(r\Sigma^2-Ma^2\sin^2\theta(r^2-a^2\cos^2\theta))}{\Sigma^2}\right)\bigg) \\
&= -\frac{\alpha Ma\sin^2\theta}{\Sigma^3\triangle}\bigg(\Big((r^2+a^2)^2-(r^2-2Mr+a^2)a^2\sin^2\theta\Big)\;(r^2-a^2\cos^2\theta) \\
&\quad +2r\Big(r\Sigma^2-Ma^2\sin^2\theta(r^2-a^2\cos^2\theta)\Big)\bigg) \\
&= -\frac{\alpha Ma\sin^2\theta}{\Sigma^3\triangle}\bigg(2r^2\Sigma^2 \\
&\quad +\Big((r^2+a^2)^2-(r^2+a^2)a^2\sin^2\theta\Big)\;(r^2-a^2\cos^2\theta)\bigg) \\
&= -\frac{\alpha Ma\sin^2\theta\;\;(3r^6+r^4a^2+4r^4a^2\cos^2\theta+r^2a^4\cos^4\theta-a^6\cos^4\theta)}{\Sigma^3\triangle} \\
&= -\frac{\alpha Ma\sin^2\theta\;\;(r^2(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta)+\cancel{r^2a^4\cos^2\theta})}{\Sigma^3\triangle} \\
&\quad -\frac{\alpha Ma\sin^2\theta\;\;(a^2\cos^2\theta(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta)-\cancel{r^2a^4\cos^2\theta})}{\Sigma^3\triangle} \\
&= -\frac{\alpha Ma\sin^2\theta\;\;(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta)}{\Sigma^2\triangle}\;.
\end{aligned}
\tag{A.1.30}
$$

Taking now equation (A.1.29g), and substituting for the inverse metric components (A.1.10) and the metric derivatives with respect to θ, (A.1.14):

$$
\begin{aligned}
K_{\theta\phi} &= \frac{1}{2}\alpha\Bigg(g^{tt}\left(\frac{\partial g_{\phi t}}{\partial x^{\theta}}\right) + g^{t\phi}\left(\frac{\partial g_{\phi\phi}}{\partial x^{\theta}}\right)\Bigg) \\
&= \frac{1}{2}\alpha\Bigg(-\frac{((r^2+a^2)^2-(r^2-2Mr+a^2)a^2\sin^2\theta)}{\Sigma\triangle} \\
&\qquad \left(-\frac{4rMa\sin\theta\ \cos\theta\ (r^2+a^2)}{\Sigma^2}\right) \\
&\qquad -\frac{2rMa}{\Sigma\triangle}\Bigg(\frac{2\sin\theta\ \cos\theta}{\Sigma^2} \\
&\qquad \Big((r^2+a^2)^3-(\Sigma+r^2+a^2)(r^2-2rM+a^2)a^2\sin^2\theta\ \Big)\Bigg)\Bigg) \\
&= \frac{2rMa\alpha\sin\theta\ \cos\theta}{\Sigma^3\triangle}\Bigg(\\
&\qquad \Big(\cancel{(r^2+a^2)^3} - \cancel{(r^2+a^2)(r^2-2Mr+a^2)}a^2\sin^2\theta\Big) \\
&\qquad -\Big(\cancel{(r^2+a^2)^3} - (\Sigma+\cancel{r^2+a^2})(r^2-2rM+a^2)a^2\sin^2\theta\Big)\Bigg) \\
&= \frac{2rMa\alpha\sin\theta\ \cos\theta}{\Sigma^{\cancelto{2}{3}}\cancel{\triangle}}\Big(\cancel{\Sigma}\cancel{(r^2-2rM+a^2)}a^2\sin^2\theta\Big) \\
&= \frac{2rMa^3\alpha\sin^3\theta\ \cos\theta}{\Sigma^2}\ .
\end{aligned}
\tag{A.1.31}
$$

The non-zero components of the extrinsic curvature for the Kerr metric (A.1.1) in Boyer-Lindquist coordinates, in the terms used in the metric, are given from equations (A.1.30) and (A.1.31) by:

$$K_{r\phi} = -\frac{\alpha M a \sin^2\theta \ \left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\triangle}, \tag{A.1.32a}$$

$$K_{\theta\phi} = \frac{2rMa^3\alpha\sin^3\theta \ \cos\theta}{\Sigma^2}, \tag{A.1.32b}$$

though the lapse is still given by α, and can be evaluated from equation (A.1.16).

To simplify further, approximations can be taken for large values of r, giving the behavior of the curvature far from the source:

$$\begin{aligned} K_{r\phi} &= -\frac{\alpha M a \sin^2\theta \ \left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\triangle} \\ &\simeq -\frac{3Ma\sin^2\theta}{r^2}, \end{aligned} \tag{A.1.33a}$$

$$\begin{aligned} K_{\theta\phi} &= \frac{2rMa^3\alpha\sin^3\theta \ \cos\theta}{\Sigma^2} \\ &\simeq \frac{2Ma^3\sin^3\theta \ \cos\theta}{r^3}, \end{aligned} \tag{A.1.33b}$$

noting that the lapse α approaches 1 for large values of r.

Far from the source, the extrinsic curvature for the Kerr metric can be given, in matrix form, by:

$$K_{ab} \simeq \begin{pmatrix} 0 & 0 & -\frac{3Ma\sin^2\theta}{r^2} \\ 0 & 0 & \frac{2Ma^3\sin^3\theta \ \cos\theta}{r^3} \\ -\frac{3Ma\sin^2\theta}{r^2} & \frac{2Ma^3\sin^3\theta \ \cos\theta}{r^3} & 0 \end{pmatrix}, \tag{A.1.34}$$

showing the curvature to be given simply, in terms of the r and θ coordinates, the rotation factor a, and the Schwarzschild mass M.

A.1.7 Extrinsic Curvature with Raised Indices

The extrinsic curvature, with the indices raised, is given by contracting the lower curvature components (A.1.32a) and (A.1.32b) with the inverse spatial metric γ^{ab}:

$$\begin{aligned} K^{ab} &= \gamma^{ai}\gamma^{bj}K_{ij} \\ &= \gamma^{ar}\gamma^{b\phi}K_{r\phi} + \gamma^{a\phi}\gamma^{br}K_{\phi r} + \gamma^{a\theta}\gamma^{b\phi}K_{\theta\phi} + \gamma^{a\phi}\gamma^{b\theta}K_{\phi\theta} \; . \end{aligned} \tag{A.1.35}$$

However, since the spatial metric γ_{ab} is diagonal, the inverse terms are simply the reciprocal of the γ_{ab} terms, which can be seen by (A.1.15) to be given by the spatial components of (A.1.3). Hence γ^{ab} is also diagonal, and the non-zero extrinsic curvature components, with raised indices, are given by:

$$\begin{aligned} K^{\phi r} \;=\; K^{r\phi} &= \gamma^{rr}\gamma^{\phi\phi}K_{r\phi} \\ &= \frac{\cancel{\triangle}}{\cancel{\Sigma}}\left(\frac{\cancel{\Sigma}}{\cancel{\sin^2\theta}\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\right) \frac{-\alpha M a \cancel{\sin^2\theta}\;\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\cancel{\triangle}} \\ &= \frac{-\alpha M a\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \; , \end{aligned} \tag{A.1.36a}$$

$$\begin{aligned} K^{\phi\theta} \;=\; K^{\theta\phi} &= \gamma^{\theta\theta}\gamma^{\phi\phi}K_{\theta\phi} \\ &= \frac{1}{\cancel{\Sigma}}\left(\frac{\cancel{\Sigma}}{\cancel{\sin^2\theta}\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\right) \frac{2rMa^3\alpha\sin^{\cancel{3}}\theta\;\cos\theta}{\Sigma^2} \\ &= \frac{2rMa^3\alpha\sin\theta\;\cos\theta}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \; . \end{aligned} \tag{A.1.36b}$$

For large r, these components reduce to:

$$K^{r\phi} \simeq -\frac{3Ma}{r^4} \; , \tag{A.1.37a}$$

$$K^{\theta\phi} \simeq \frac{2Ma^3\sin\theta\;\cos\theta}{r^7} \; . \tag{A.1.37b}$$

A.2 Derivatives of Extrinsic Curvature Components

The derivatives of the raised index components of the extrinsic curvature (A.1.36a) and (A.1.36b), with respect to r and θ respectively, are given by:

$$\partial_r \; K^{\phi r} \; = \partial_r \left(\frac{-\alpha M a \left(3r^4 + r^2 a^2 + r^2 a^2 \cos^2\theta - a^4 \cos^2\theta\right)}{\Sigma^2 \left((r^2+a^2)^2 - \triangle a^2 \sin^2\theta\right)} \right) , \tag{A.2.1a}$$

$$\partial_\theta \; K^{\phi\theta} \; = \partial_\theta \left(\frac{2rMa^3 \alpha \sin\theta \; \cos\theta}{\Sigma^2 \left((r^2+a^2)^2 - \triangle a^2 \sin^2\theta\right)} \right) . \tag{A.2.1b}$$

Both curvature components can be broken into parts of the form $\frac{X}{Y} = \frac{AB}{CD}$, and the derivatives then found using the quotient, product and chain rules:

$$\partial \left(\frac{AB}{CD} \right) \; = \partial \left(\frac{X}{Y} \right) \; = \; \frac{Y \; \partial X - X \; \partial Y}{Y^2} \; , \tag{A.2.2a}$$

$$\partial X \; = \partial(AB) \; = \; A \; \partial B + B \; \partial A \; , \tag{A.2.2b}$$

$$\partial Y \; = \partial(CD) \; = \; C \; \partial D + D \; \partial C \; , \tag{A.2.2c}$$

$$\begin{aligned} \Leftrightarrow \qquad \partial \left(\frac{AB}{CD} \right) \; &= \frac{CD \; (A \; \partial B + B \; \partial A) - AB \; (C \; \partial D + D \; \partial C)}{C^2 D^2} \\ &= \frac{(A \; \partial B + B \; \partial A)}{CD} - \frac{AB \; (C \; \partial D + D \; \partial C)}{C^2 D^2} \\ &= \frac{A \; \partial B}{CD} + \frac{B \; \partial A}{CD} - \frac{AB \; \partial D}{CD^2} - \frac{AB \; \partial C}{C^2 D} \; . \end{aligned} \tag{A.2.3}$$

A.2.1 Derivative of $K^{\phi r}$, with respect to r

The derivatives with respect to r, are first taken of the separate parts of $K^{\phi r}$ from (A.1.36a):

$$\begin{aligned}
&\partial_r \ \alpha \\
&= \ \partial_r \ \sqrt{\frac{\Sigma\triangle}{(r^2+a^2)^2-(r^2-2rM+a^2)a^2\sin^2\theta}} \\
&= \ \frac{1}{\cancel{2}\alpha}\left(\frac{\cancel{2}(2r^3-3r^2M+ra^2+ra^2\cos^2\theta-Ma^2\cos^2\theta)}{((r^2+a^2)^2-(r^2-2rM+a^2)a^2\sin^2\theta)}\right. \\
&\quad \left. -\frac{\Sigma\triangle\cancel{2}\,(2r^3+ra^2+ra^2\cos^2\theta+Ma^2\sin^2\theta)}{\left((r^2+a^2)^2-(r^2-2rM+a^2)a^2\sin^2\theta\right)^2}\right) \\
&= \ \left(\frac{(2r^3-3r^2M+ra^2+ra^2\cos^2\theta-Ma^2\cos^2\theta)}{\alpha\,((r^2+a^2)^2-(r^2-2rM+a^2)a^2\sin^2\theta)}\right. \\
&\quad \left. -\frac{\Sigma\triangle\,(2r^3+ra^2+ra^2\cos^2\theta+Ma^2\sin^2\theta)}{\alpha\left((r^2+a^2)^2-(r^2-2rM+a^2)a^2\sin^\theta\right)^2}\right) \ ,
\end{aligned} \tag{A.2.4a}$$

$$\begin{aligned}
&\partial_r \ \left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right) \\
&= \ 2r\left(6r^2+a^2+a^2\cos^2\theta\right) \ ,
\end{aligned} \tag{A.2.4b}$$

$$\begin{aligned}
&\partial_r \ \Sigma^2 \\
&= \ \partial_r \ (r^2+a^2\cos^2\theta)^2 \\
&= \ 4r(r^2+a^2\cos^2\theta) \ = \ 4r\Sigma \ ,
\end{aligned} \tag{A.2.4c}$$

$$\begin{aligned}
&\partial_r \ \left((r^2+a^2)^2-(r^2-2rM+a^2)a^2\sin^2\theta\right) \\
&= \ \left(4r(r^2+a^2)-(2r-2M)a^2\sin^2\theta\right) \\
&= \ 2\left(2r^3+2ra^2-ra^2\sin^2\theta+Ma^2\sin^2\theta\right) \\
&= \ 2\left(2r^3+ra^2+ra^2\cos^2\theta+Ma^2\sin^2\theta\right) \ .
\end{aligned} \tag{A.2.4d}$$

Using the derivative rules from (A.2.3), and the derivatives of the separate parts of $K^{\phi r}$ from (A.2.4), the derivative $\partial_r\ K^{\phi r}$ from (A.2.1a) is given by:

$$\begin{aligned}
\partial_r\ K^{\phi r} \ =\ & -\frac{\alpha M a\ 2r\left(6r^2+a^2+a^2\cos^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)} \\
& -\frac{Ma\ \left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)} \\
& \Bigg(\frac{\left(2r^3-3r^2M+ra^2+ra^2\cos^2\theta-Ma^2\cos^2\theta\right)}{\alpha\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)} \\
& \quad -\frac{\Sigma\triangle\left(2r^3+ra^2+ra^2\cos^2\theta+Ma^2\sin^2\theta\right)}{\alpha\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)^2}\Bigg) \qquad \text{(A.2.5)} \\
& +\frac{\alpha Ma\ \left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)^2} \\
& 2\left(2r^3+ra^2+ra^2\cos^2\theta+Ma^2\sin^2\theta\right) \\
& +\frac{\alpha Ma\ \left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right)4r\cancel{\Sigma}}{\Sigma^{\cancelto{3}{4}}\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}\ ,
\end{aligned}$$

and simplifying the equation, by taking out common terms:

$$\begin{aligned}
\partial_r\ K^{\phi r} \ =\ & \frac{\alpha Ma}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)} \\
& \Bigg(-2r\left(6r^2+a^2+a^2\cos^2\theta\right) \\
& -\frac{\left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right)}{\alpha^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)} \\
& \Bigg(\frac{\Sigma\triangle(2r^3-3r^2M+ra^2+ra^2\cos^2\theta-Ma^2\cos^2\theta)}{\Sigma\triangle} \qquad \text{(A.2.6)} \\
& \quad -\frac{\Sigma\triangle\left(2r^3+ra^2+ra^2\cos^2\theta+Ma^2\sin^2\theta\right)}{\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}\Bigg) \\
& +\frac{2\left(2r^3+ra^2+ra^2\cos^2\theta+Ma^2\sin^2\theta\right)}{\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)} \\
& \left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right) \\
& +\frac{4r\left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right)}{\Sigma}\Bigg)\ .
\end{aligned}$$

Substituting in for the lapse squared terms, from equation (A.1.16), and simplifying further:

$$\begin{aligned}
\partial_r \; K^{\phi r} \;\; &= \frac{\alpha M a}{\Sigma^2 \left((r^2+a^2)^2 - \triangle a^2 \sin^2\theta\right)} \\
&\Bigg(- 2r \left(6r^2 + a^2 + a^2 \cos^2\theta\right) \\
&- \left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right) \\
&\Bigg(\frac{(2r^3 - 3r^2M + ra^2 + ra^2\cos^2\theta - Ma^2\cos^2\theta)}{\Sigma\triangle} \\
&- \frac{(2r^3 + ra^2 + ra^2\cos^2\theta + Ma^2\sin^2\theta)}{((r^2+a^2)^2 - \triangle a^2\sin^2\theta)}\Bigg) \\
&+ \frac{2\,(2r^3 + ra^2 + ra^2\cos^2\theta + Ma^2\sin^2\theta)}{((r^2+a^2)^2 - \triangle a^2\sin^2\theta)} \\
&\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right) \\
&+ \frac{4r\,(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta)}{\Sigma}\Bigg) \; ,
\end{aligned} \tag{A.2.7}$$

and grouping similar parts together and simplifying again:

$$\begin{aligned}
\partial_r \; K^{\phi r} \;\; &= \frac{\alpha M a}{\Sigma^2 \left((r^2+a^2)^2 - \triangle a^2 \sin^2\theta\right)} \\
&\Bigg(- 2r \left(6r^2 + a^2 + a^2 \cos^2\theta\right) \\
&+ \frac{4r\triangle - (2r^3 - 3r^2M + ra^2 + ra^2\cos^2\theta - Ma^2\cos^2\theta)}{\Sigma\triangle} \\
&\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right) \\
&+ \frac{3\,(2r^3 + ra^2 + ra^2\cos^2\theta + Ma^2\sin^2\theta)}{((r^2+a^2)^2 - \triangle a^2\sin^2\theta)} \\
&\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)\Bigg) \; .
\end{aligned} \tag{A.2.8}$$

The r derivative of the extrinsic curvature component $K^{\phi r}$, can finally be given in the reduced form by:

$$\begin{aligned}
\partial_r\ K^{\phi r} \ &= \frac{\alpha M a}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)} \\
&\Bigg(-2r\left(6r^2+a^2+a^2\cos^2\theta\right) \\
&+\frac{(2r^3-5r^2M+3ra^2-ra^2\cos^2\theta+Ma^2\cos^2\theta)}{\Sigma\triangle} \\
&\left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right) \\
&+\frac{3\left(2r^3+ra^2+ra^2\cos^2\theta+Ma^2\sin^2\theta\right)}{\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)} \\
&\left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right)\Bigg)\ .
\end{aligned} \tag{A.2.9}$$

A.2.2 Derivative of $K^{\phi\theta}$, with respect to θ

The derivatives of the separate parts of the component $K^{\phi\theta}$ from (A.1.36b), are now taken, with respect to θ:

$$\begin{aligned}
&\partial_\theta\ \alpha \\
&\quad = \partial_\theta \sqrt{\frac{\Sigma\triangle}{(r^2+a^2)^2-(r^2-2rM+a^2)a^2\sin^2\theta}} \\
&\quad = \frac{1}{2\alpha}\Bigg(\frac{-2a^2\cos\theta\ \sin\theta\ \triangle}{((r^2+a^2)^2-(r^2-2rM+a^2)a^2\sin^2\theta)} \\
&\qquad\quad - \frac{-2(r^2-2rM+a^2)a^2\cos\theta\ \sin\theta\ \Sigma\triangle}{\left((r^2+a^2)^2-(r^2-2rM+a^2)a^2\sin^2\theta\right)^2}\Bigg) \\
&\quad = \Bigg(\frac{-a^2\cos\theta\ \sin\theta\ \triangle}{\alpha\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)} \\
&\qquad\quad + \frac{a^2\cos\theta\ \sin\theta\ \Sigma\triangle^2}{\alpha\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)^2}\Bigg)\ ,
\end{aligned} \tag{A.2.10a}$$

$$\begin{aligned}
&\partial_\theta\ \sin\theta\ \cos\theta \\
&\quad = \left(\cos^2\theta-\sin^2\theta\right)\ ,
\end{aligned} \tag{A.2.10b}$$

$$\begin{aligned}
&\partial_\theta\ \Sigma^2 \\
&\quad = \partial_\theta\ \left(r^2+a^2\cos^2\theta\right)^2 \\
&\quad = -4a^2\cos\theta\ \sin\theta\ \left(r^2+a^2\cos^2\theta\right) \\
&\quad = -4a^2\cos\theta\ \sin\theta\ \Sigma\ ,
\end{aligned} \tag{A.2.10c}$$

$$\begin{aligned}
&\partial_\theta\ \left((r^2+a^2)^2-(r^2-2rM+a^2)a^2\sin^2\theta\right) \\
&\quad = -2(r^2-2rM+a^2)a^2\cos\theta\ \sin\theta\ .
\end{aligned} \tag{A.2.10d}$$

Using the derivative rules from (A.2.3), and the derivatives of the separate parts of $K^{\phi r}$ from (A.2.10), the derivative $\partial_r\ K^{\phi r}$ from (A.2.1b) is given by:

$$\begin{aligned}
\partial_\theta\ K^{\phi\theta} \ &= \frac{2rMa^3\alpha\ \left(\cos^2\theta - \sin^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&+ \frac{2rMa^3\cos\theta\ \sin\theta}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&\left(\frac{-a^2\cos\theta\ \sin\theta\ \triangle}{\alpha\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\right. \\
&\quad \left. + \frac{a^2\cos\theta\ \sin\theta\ \Sigma\triangle^2}{\alpha\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)^2}\right) \\
&- \frac{-2rMa^3\alpha\cos\theta\ \sin\theta\ 2(r^2-2rM+a^2)a^2\cos\theta\ \sin\theta}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)^2} \\
&- \frac{-2rMa^3\alpha\cos\theta\ \sin\theta\ 4a^2\cos\theta\ \sin\theta\ \Sigma}{\Sigma^4\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\ ,
\end{aligned} \tag{A.2.11}$$

and simplifying the equation, by taking out common terms:

$$\begin{aligned}
\partial_\theta\ K^{\phi\theta} \ &= \frac{2r\alpha Ma^3}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&\left(\left(\cos^2\theta - \sin^2\theta\right)\right. \\
&\quad + \frac{\Sigma\triangle}{\alpha^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&\quad \left(\frac{-a^2\cos^2\theta\ \sin^2\theta}{\Sigma} + \frac{a^2\cos^2\theta\ \sin^2\theta\ \triangle}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\right) \\
&\quad \left. + \frac{2\triangle a^2\cos^2\theta\ \sin^2\theta}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} + \frac{4a^2\cos^2\theta\ \sin^2\theta}{\Sigma}\right)\ .
\end{aligned} \tag{A.2.12}$$

Substituting in for the lapse squared terms, from equation (A.1.16), and simplifying further:

$$
\begin{aligned}
\partial_\theta\ K^{\phi\theta} \ &= \frac{2r\alpha M a^3}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2 \sin^2\theta\right)} \\
&\left(\cos^2\theta - \sin^2\theta \right.\\
&\quad - \frac{a^2\cos^2\theta\ \sin^2\theta}{\Sigma} + \frac{a^2\triangle\cos^2\theta\ \sin^2\theta}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&\quad \left. + \frac{2\triangle a^2\cos^2\theta\ \sin^2\theta}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \ + \ \frac{4a^2\cos^2\theta\ \sin^2\theta}{\Sigma}\right) .
\end{aligned}
\tag{A.2.13}
$$

The θ derivative of the extrinsic curvature component $K^{\phi\theta}$, can finally be given in the reduced form by:

$$
\begin{aligned}
\partial_\theta\ K^{\phi\theta} \ &= \frac{2r\alpha M a^3}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2 \sin^2\theta\right)} \\
&\left(\cos^2\theta - \sin^2\theta \ + \ \frac{3a^2\cos^2\theta\ \sin^2\theta}{\Sigma} \ + \ \frac{3a^2\triangle\cos^2\theta\ \sin^2\theta}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\right) .
\end{aligned}
\tag{A.2.14}
$$

A.3 Einstein Constraints

If the extrinsic curvature has been calculated properly, it must satisfy the constraint equations of the 3+1 formalism, since the Kerr metric is *known* to be an exact solution. The constraints are therefore used to verify that the calculations have been carried out properly, and that equations (A.1.36a) and (A.1.36b) *do* represent the non-zero components of the extrinsic curvature K^{ab} for the Kerr metric (A.1.1).

A.3.1 Hamiltonian Constraint

The Hamiltonian constraint is given from equation (2.1.27), by:

$$ {}^{(3)}R + K^2 - K_{ab}\, K^{ab} \;=\; 16\pi\rho \; . \tag{A.3.1}$$

Since the Kerr metric describes a vacuum, $\rho = 0$, and it can be seen that the trace of the extrinsic curvature is zero, the Hamiltonian constraint reduces to:

$$ {}^{(3)}R \;=\; K_{ab}\, K^{ab} \; . \tag{A.3.2}$$

The 3-space scalar curvature ${}^{(3)}R$, from equations (1.2.72), (1.2.71) and (1.2.70), is given by:

$$\begin{aligned} {}^{(3)}R &= \gamma^{ab}\; {}^{(3)}R_{ab} \\ &= \gamma^{ab}\; {}^{(3)}R^{c}{}_{acb} \\ &= \gamma^{ab}\; \Big(\partial_c \Gamma^{c}{}_{ab} \;-\; \partial_a \Gamma^{c}{}_{cb} \;+\; \Big(\; \Gamma^{d}{}_{ab}\Gamma^{c}{}_{dc} - \Gamma^{d}{}_{cb}\Gamma^{c}{}_{da} \;\Big)\Big) \\ &= \gamma^{aa}\; \Big(\partial_c \Gamma^{c}{}_{aa} \;-\; \partial_a \Gamma^{c}{}_{ca} \;+\; \Big(\; \Gamma^{d}{}_{aa}\Gamma^{c}{}_{dc} - \Gamma^{d}{}_{ca}\Gamma^{c}{}_{da} \;\Big)\Big) \; , \end{aligned} \tag{A.3.3}$$

using the fact that γ_{ab}, and its inverse, are diagonal matrices, in the final line above.

The scalar curvature (A.3.3) was calculated for the Kerr metric (A.1.1), using *Mathematica*, giving the solution:

$$ {}^{(3)}R \;\simeq\; \frac{18 M^2 a^2 \sin^2\theta}{r^6} \; , \tag{A.3.4}$$

in the approximation as r tends to infinity.

A.3.2 Scalar Form of Extrinsic Curvature

The scalar product of the extrinsic curvature $K_{ab}K^{ab}$, is now calculated from the curvature components (A.1.32) and (A.1.36):

$$
\begin{aligned}
K_{ab}K^{ab} &= K_{r\phi}K^{r\phi} + K_{\phi r}K^{\phi r} + K_{\theta\phi}K^{\theta\phi} + K_{\phi\theta}K^{\phi\theta} \\
&= 2K_{r\phi}K^{r\phi} \;+\; 2K_{\theta\phi}K^{\theta\phi} \\
&= 2\frac{-\alpha M a \sin^2\theta \;\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\triangle} \\
&\quad \frac{-\alpha M a \left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&\quad + \;2\;\frac{2rMa^3\alpha\sin^3\theta\;\cos\theta}{\Sigma^2}\frac{2rMa^3\alpha\sin\theta\;\cos\theta}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&= \frac{2\alpha^2M^2a^2\sin^2\theta}{\Sigma^4\triangle\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&\quad \left(\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)^2 + 4r^2a^4\sin^2\theta\;\cos^2\theta\triangle\right) \\
&= \frac{2\alpha^2M^2a^2\sin^2\theta}{\Sigma^4(r^2-2rM+a^2)\left((r^2+a^2)^2 - (r^2-2rM+a^2)a^2\sin^2\theta\right)} \\
&\quad \left(9r^8 + 6r^6a^2 + 6r^6a^2\cos^2\theta + r^4a^4 - 3r^4a^4\cos^4\theta - 8r^3Ma^4\cos^2\theta\;\sin^2\theta\right. \\
&\quad \left. + 4r^2a^6\cos^2\theta + 2r^2a^6\cos^2\theta\;\sin^2\theta + a^8\cos^4\theta\right)\;,
\end{aligned}
\tag{A.3.5}
$$

and approximating the result above as r tends to infinity:

$$
K_{ab}K^{ab} \;\simeq\; \frac{18\alpha^2M^2a^2\sin^2\theta}{r^6} \tag{A.3.6}
$$

and with the lapse α approximating to a unit timelike normal as r tends to infinity:

$$
K_{ab}K^{ab} \;\simeq\; \frac{18M^2a^2\sin^2\theta}{r^6} \tag{A.3.7}
$$

which is exactly the 3-space scalar curvature ${}^{(3)}R$, from (A.3.4), thereby showing the Hamiltonian constraint (A.3.2) to be satisfied, at least for large values of r.

A.3.3 Momentum Constraint

The momentum constraint is given by equation (2.1.30):

$$D_b\left(K^{ab} - \gamma^{ab}K\right) \;=\; -8\pi j^a \;, \tag{A.3.8}$$

and again, since the Kerr metric describes a vacuum, $j^a = 0$, and the trace of the extrinsic curvature is zero, the momentum constraint reduces to:

$$D_b K^{ab} \;=\; 0 \;, \tag{A.3.9}$$

showing the extrinsic curvature of Kerr, in Boyer-Lindquist coordinates, to be given by an axially-symmetric transverse trace-free tensor.

The divergence of K^{ab} is first given with respect to an arbitrary diagonal 3-space metric γ_{ab}:

$$\begin{aligned}
D_b K^{ab} &= \partial_b K^{ab} + \Gamma^a_{bc}K^{cb} + \Gamma^b_{bc}K^{ac} \\[2ex]
&= \partial_b K^{ab} + \frac{1}{2}\gamma^{ad}\left(\partial_c\gamma_{bd} + \partial_b\gamma_{cd} - \partial_d\gamma_{bc}\right)K^{cb} \\
&\quad + \frac{1}{2}\gamma^{bd}\left(\partial_c\gamma_{bd} + \partial_b\gamma_{cd} - \partial_d\gamma_{bc}\right)K^{ac} \\[2ex]
&= \partial_b K^{ab} + \frac{1}{2}\gamma^{aa}\left(\partial_c\gamma_{ba} + \partial_b\gamma_{ca} - \partial_a\gamma_{bc}\right)K^{cb} \\
&\quad + \frac{1}{2}\gamma^{bb}\left(\partial_c\gamma_{bb} + \cancel{\partial_b\gamma_{cb}} - \cancel{\partial_b\gamma_{bc}}\right)K^{ac} \\[2ex]
&= \partial_b K^{ab} + \frac{1}{2}\gamma^{aa}\left(\partial_c\gamma_{aa}K^{ca} + \partial_b\gamma_{aa}K^{ab} - \partial_a\gamma_{bb}\cancelto{0}{K^{bb}}\right) + \frac{1}{2}\gamma^{bb}\partial_c\gamma_{bb}K^{ac} \\[2ex]
&= \partial_b K^{ab} + \gamma^{aa}\partial_b\gamma_{aa}K^{ab} + \frac{1}{2}\gamma^{bb}\partial_c\gamma_{bb}K^{ac} \;,
\end{aligned} \tag{A.3.10}$$

with the c indices in the brackets of the second last line changed to b, since both indices are summed over, and there are no other b indices interacting with this term.

Breaking the divergence $D_b K^{ab}$, from equation (A.3.10), into it's three separate components, and imposing conditions from the Kerr metric components (A.1.3), and extrinsic curvature components (A.1.36):

$$
\begin{aligned}
D_b K^{rb} &= \partial_b K^{rb} + \gamma^{rr}\ \partial_b \gamma_{rr}\ K^{rb} + \frac{1}{2}\ \gamma^{bb}\ \partial_c \gamma_{bb}\ K^{rc} \\
&= \cancelto{0}{\partial_\phi K^{r\phi}} + \gamma^{rr}\ \cancelto{0}{\partial_\phi \gamma_{rr}}\ K^{r\phi} + \frac{1}{2}\ \gamma^{bb}\ \cancelto{0}{\partial_\phi \gamma_{bb}}\ K^{r\phi} \\
&= 0\ ,
\end{aligned}
\tag{A.3.11}
$$

$$
\begin{aligned}
D_b K^{\theta b} &= \partial_b K^{\theta b} + \gamma^{\theta\theta}\ \partial_b \gamma_{\theta\theta}\ K^{\theta b} + \frac{1}{2}\gamma^{bb}\ \partial_c \gamma_{bb}\ K^{\theta c} \\
&= \cancelto{0}{\partial_\phi K^{\theta\phi}} + \gamma^{\theta\theta}\ \cancelto{0}{\partial_\phi \gamma_{\theta\theta}}\ K^{\theta\phi} + \frac{1}{2}\ \gamma^{bb}\ \cancelto{0}{\partial_\phi \gamma_{bb}}\ K^{\theta\phi} \\
&= 0\ ,
\end{aligned}
\tag{A.3.12}
$$

$$
\begin{aligned}
D_b K^{\phi b} &= \partial_b K^{\phi b} + \gamma^{\phi\phi}\ \partial_b \gamma_{\phi\phi}\ K^{\phi b} + \frac{1}{2}\ \gamma^{bb}\ \partial_c \gamma_{bb}\ K^{\phi c} \\
\\
&= \partial_r K^{\phi r} + \gamma^{\phi\phi}\ \partial_r \gamma_{\phi\phi}\ K^{\phi r} + \frac{1}{2}\ \gamma^{bb}\ \partial_r \gamma_{bb}\ K^{\phi r} \\
&\quad + \partial_\theta K^{\phi\theta} + \gamma^{\phi\phi}\ \partial_\theta \gamma_{\phi\phi}\ K^{\phi\theta} + \frac{1}{2}\ \gamma^{bb}\ \partial_\theta \gamma_{bb}\ K^{\phi\theta} \\
\\
&= \partial_r K^{\phi r}\ +\ \partial_\theta K^{\phi\theta} \\
&\quad +\ \gamma^{\phi\phi}\ K^{\phi r} \partial_r \gamma_{\phi\phi}\ +\ \gamma^{\phi\phi}\ K^{\phi\theta} \partial_\theta \gamma_{\phi\phi} \\
&\quad + \frac{1}{2}\ \gamma^{rr}\ K^{\phi r} \partial_r \gamma_{rr} + \frac{1}{2}\ \gamma^{rr}\ K^{\phi\theta} \partial_\theta \gamma_{rr} \\
&\quad + \frac{1}{2}\ \gamma^{\theta\theta}\ K^{\phi r}\ \partial_r \gamma_{\theta\theta} + \frac{1}{2}\ \gamma^{\theta\theta}\ K^{\phi\theta}\ \partial_\theta \gamma_{\theta\theta} \\
&\quad + \frac{1}{2}\ \gamma^{\phi\phi}\ K^{\phi r}\ \partial_r \gamma_{\phi\phi} + \frac{1}{2}\ \gamma^{\phi\phi}\ K^{\phi\theta}\ \partial_\theta \gamma_{\phi\phi} \\
\\
&= \partial_r\ K^{\phi r} +\ \partial_\theta\ K^{\phi\theta} \\
&\quad +\ \frac{1}{2}\ \left(\gamma^{rr}\ \partial_r \gamma_{rr}\ +\ \gamma^{\theta\theta}\ \partial_r \gamma_{\theta\theta}\ +\ 3\ \gamma^{\phi\phi}\ \partial_r \gamma_{\phi\phi}\right)\ K^{\phi r} \\
&\quad +\ \frac{1}{2}\ \left(\gamma^{rr}\ \partial_\theta \gamma_{rr}\ +\ \gamma^{\theta\theta}\ \partial_\theta \gamma_{\theta\theta}\ +\ 3\ \gamma^{\phi\phi}\ \partial_\theta \gamma_{\phi\phi}\right)\ K^{\phi\theta}\ ,
\end{aligned}
\tag{A.3.13}
$$

using, in particular, the fact that none of the metric components depend on the ϕ coordinate.

Divergence Connection Coefficients for $K^{\phi r}$

The two bracketed terms of the ϕ part of the extrinsic curvature divergence $D_b K^{\phi b}$, from equation (A.3.13), are first treated separately. Substituting for the inverse spatial metric components (A.1.10), the metric derivatives (A.1.13) and (A.1.14), and $K^{\phi r}$ from (A.1.36a), the first bracketed term of (A.3.13) becomes:

$$\frac{1}{2}\left(\gamma^{rr}\;\partial_r\gamma_{rr}\;+\;\gamma^{\theta\theta}\;\partial_r\gamma_{\theta\theta}\;+\;3\;\gamma^{\phi\phi}\;\partial_r\gamma_{\phi\phi}\right)\;K^{\phi r}$$

$$= \frac{-\alpha M a\left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}$$

$$\frac{1}{2}\left(\frac{\cancel{(r^2-2rM+a^2)}}{\Sigma}\;\frac{2(-r^2M+ra^2\sin^2\theta+Ma^2\cos^2\theta)}{(r^2-2rM+a^2)^{\cancel{2}}}\right.$$

$$+\;\frac{1}{\Sigma}\;(2r)$$

$$+\;3\;\frac{\cancel{\Sigma}}{\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)\cancel{\sin^2\theta}}$$

$$\left.\frac{2\cancel{\sin^2\theta}}{\Sigma^{\cancel{2}}}\left(\Sigma(2r^3+ra^2+ra^2\cos^2\theta+Ma^2\sin^2\theta)-r\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)\right)\right)\,. \tag{A.3.14}$$

Equation (A.3.14) is now simplified further:

$$\frac{1}{2}\left(\gamma^{rr}\,\frac{\partial\gamma_{rr}}{\partial x^r} \;+\; \gamma^{\theta\theta}\,\frac{\partial\gamma_{\theta\theta}}{\partial x^r} \;+\; 3\,\gamma^{\phi\phi}\,\frac{\partial\gamma_{\phi\phi}}{\partial x^r}\right)\,K^{\phi r}$$

$$\begin{aligned}
&= \frac{-\alpha M a\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&\quad\left(\frac{(-r^2M + ra^2\sin^2\theta + Ma^2\cos^2\theta)}{\Sigma(r^2-2rM+a^2)} \;+\; \frac{r}{\Sigma}\right. \\
&\quad -\;\frac{3r\cancel{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}}{\Sigma\cancel{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}} \\
&\quad \left. +\;\frac{3\cancel{\Sigma}(2r^3 + ra^2 + ra^2\cos^2\theta + Ma^2\sin^2\theta)}{\cancel{\Sigma}\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\right) \\
&= \frac{\alpha M a\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&\quad\left(\frac{(r^2M - ra^2\sin^2\theta - Ma^2\cos^2\theta)}{\Sigma(r^2-2rM+a^2)} \;+\; \frac{2r}{\Sigma}\right. \\
&\quad \left. -\;\frac{3(2r^3 + ra^2 + ra^2\cos^2\theta + Ma^2\sin^2\theta)}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\right) \\
&= \frac{\alpha M a\left(3r^4 + r^2a^2 + r^2a^2\cos^2\theta - a^4\cos^2\theta\right)}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)} \\
&\quad\left(\frac{2r^3 - 3r^2M + ra^2 + ra^2\cos^2\theta - Ma^2\cos^2\theta}{\Sigma(r^2-2rM+a^2)}\right. \\
&\quad \left. -\;\frac{3(2r^3 + ra^2 + ra^2\cos^2\theta + Ma^2\sin^2\theta)}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}\right),
\end{aligned} \tag{A.3.15}$$

giving the first bracketed term of (A.3.13), reduced as much as possible.

Divergence Connection Coefficients for $K^{\phi\theta}$

Substituting for the inverse spatial metric components (A.1.10), the metric derivatives (A.1.13) and (A.1.14), and $K^{\phi r}$ from (A.1.36a), the *second* bracketed term of (A.3.13) becomes:

$$\frac{1}{2}\left(\gamma^{rr}\,\frac{\partial\gamma_{rr}}{\partial x^{\theta}} \;+\; \gamma^{\theta\theta}\,\frac{\partial\gamma_{\theta\theta}}{\partial x^{\theta}} \;+\; 3\,\gamma^{\phi\phi}\,\frac{\partial\gamma_{\phi\phi}}{\partial x^{\theta}}\right)\,K^{\phi\theta}$$

$$\begin{aligned}
&= \frac{2rMa^3\alpha\sin\theta\,\cos\theta}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)} \\
&\quad \frac{1}{2}\left(\frac{\cancel{(r^2-2rM+a^2)}}{\Sigma}\,\frac{-2a^2\cos\theta\,\sin\theta}{\cancel{(r^2-2rM+a^2)}} \;+\; \frac{1}{\Sigma}\,(-2a^2\cos\theta\,\sin\theta)\right. \\
&\quad +\;3\,\frac{\cancel{\Sigma}}{\left((r^2+a^2)^2-(r^2-2rM+a^2)a^2\sin^2\theta\right)\sin^{\cancel{2}}\theta} \\
&\quad \left.\frac{2\cancel{\sin\theta}\,\cos\theta}{\Sigma^{\cancel{2}}}\left((r^2+a^2)^3-(\Sigma+r^2+a^2)(r^2-2rM+a^2)a^2\sin^2\theta\right)\right)\,.
\end{aligned} \tag{A.3.16}$$

Equation (A.3.16) is now simplified further:

$$\frac{1}{2}\left(\gamma^{rr}\,\frac{\partial\gamma_{rr}}{\partial x^{\theta}}\;+\;\gamma^{\theta\theta}\,\frac{\partial\gamma_{\theta\theta}}{\partial x^{\theta}}\;+\;3\,\gamma^{\phi\phi}\,\frac{\partial\gamma_{\phi\phi}}{\partial x^{\theta}}\right)\,K^{\phi\theta}$$

$$= \frac{2rMa^3\alpha\sin\theta\;\cos\theta}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}$$

$$\left(\frac{-2a^2\cos\theta\;\sin\theta}{\Sigma}\right.$$

$$\left.+\;\frac{3\cos\theta\left((r^2+a^2)\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)-\Sigma\triangle a^2\sin^2\theta\right)}{\Sigma\sin\theta\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}\right)$$

$$= \frac{2r\alpha Ma^3}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}$$

$$\left(\frac{-2a^2\cos^2\theta\;\sin^2\theta}{\Sigma}\;+\;\frac{3\cos^2\theta(r^2+a^2)}{\Sigma}\;-\;\frac{3\triangle a^2\cos^2\theta\sin^2\theta}{\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}\right)$$

$$= \frac{2r\alpha Ma^3}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}$$

$$\left(\frac{3r^2\cos^2\theta-2a^2\cos^2\theta\;\sin^2\theta+3a^2\cos^2\theta}{\Sigma}\;-\;\frac{3\triangle a^2\cos^2\theta\sin^2\theta}{\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}\right), \tag{A.3.17}$$

giving the second bracketed term of (A.3.13), reduced as much as possible.

A.3.4 Evaluation of ϕ Divergence Term

The ϕ term of the divergence of K^{ab}, given by equation (A.3.13), is separated according to the terms depending on each of the curvature components. Each part is then evaluated separately, before being combined:

The $K^{\phi r}$ terms are taken first, with the component derivative given by equation (A.2.9), and the connection terms from (A.3.15):

$$\partial_r \; K^{\phi r} \; + \; \frac{1}{2} \; \left(\gamma^{rr} \; \frac{\partial \gamma_{rr}}{\partial x^r} \; + \; \gamma^{\theta\theta} \; \frac{\partial \gamma_{\theta\theta}}{\partial x^r} \; + \; 3 \; \gamma^{\phi\phi} \; \frac{\partial \gamma_{\phi\phi}}{\partial x^r} \right) \; K^{\phi r}$$

$$\begin{aligned}
&= \frac{\alpha M a}{\Sigma^2 \left((r^2 + a^2)^2 - \triangle a^2 \sin^2 \theta \right)} \\
&\Bigg(\Bigg(- 2r \left(6r^2 + a^2 + a^2 \cos^2 \theta \right) \\
&\quad + \left(3r^4 + r^2 a^2 + r^2 a^2 \cos^2 \theta - a^4 \cos^2 \theta \right) \\
&\quad \Bigg(\frac{(2r^3 - 5r^2 M + 3ra^2 - \cancel{ra^2 \cos^2 \theta} + \cancel{Ma^2 \cos^2 \theta})}{\Sigma \triangle} \\
&\quad + \frac{3 \left(2r^3 + ra^2 + ra^2 \cos^2 \theta \cancel{+ Ma^2 \sin^2 \theta} \right)}{\cancel{\left((r^2 + a^2)^2 - \triangle a^2 \sin^2 \theta \right)}} \Bigg) \Bigg) \\
&\quad + \left(3r^4 + r^2 a^2 + r^2 a^2 \cos^2 \theta - a^4 \cos^2 \theta \right) \\
&\quad \Bigg(\frac{2r^3 - 3r^2 M + ra^2 + \cancel{ra^2 \cos^2 \theta} - \cancel{Ma^2 \cos^2 \theta}}{\Sigma (r^2 - 2rM + a^2)} \\
&\quad - \frac{3(2r^3 + ra^2 + ra^2 \cos^2 \theta \cancel{+ Ma^2 \sin^2 \theta})}{\cancel{((r^2 + a^2)^2 - \triangle a^2 \sin^2 \theta)}} \Bigg) \Bigg) \; .
\end{aligned} \tag{A.3.18}$$

Multiplying out the individual terms of (A.3.18):

$$\partial_r\ K^{\phi r}\ +\ \frac{1}{2}\ \left(\gamma^{rr}\ \frac{\partial\gamma_{rr}}{\partial x^r}\ +\ \gamma^{\theta\theta}\ \frac{\partial\gamma_{\theta\theta}}{\partial x^r}\ +\ 3\ \gamma^{\phi\phi}\ \frac{\partial\gamma_{\phi\phi}}{\partial x^r}\right)\ K^{\phi r}$$

$$= \frac{\alpha M a}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}$$
$$\left(-2r\left(6r^2+a^2+a^2\cos^2\theta\right)\right.$$
$$\left.+\left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right)\left(\frac{(4r^3-8r^2M+4ra^2)}{\Sigma\triangle}\right)\right)$$

$$= \frac{\alpha M a}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}$$
$$\left(-2r\left(6r^2+a^2+a^2\cos^2\theta\right)+\frac{4r\left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right)}{\Sigma}\right)$$

$$= \frac{\alpha M a}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}$$
$$\left(-2r\left(6r^2+a^2+a^2\cos^2\theta\right)+\frac{4r\left(3r^4+r^2a^2+r^2a^2\cos^2\theta-a^4\cos^2\theta\right)}{\Sigma}\right)$$

$$= \frac{\alpha M a}{\Sigma^2\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}$$
$$+\frac{2r\left(r^2a^2-5r^2a^2\cos^2\theta-3a^4\cos^2\theta-a^4\cos^4\theta\right)}{\Sigma}$$

$$= \frac{2r\alpha M a^3\left(r^2-5r^2\cos^2\theta-3a^2\cos^2\theta-a^2\cos^4\theta\right)}{\Sigma^3\left((r^2+a^2)^2-\triangle a^2\sin^2\theta\right)}\ , \tag{A.3.19}$$

giving the terms for the curvature component $K^{\phi r}$, simplified as much as possible.

The $K^{\phi\theta}$ terms are now taken, with the component derivative given by equation (A.2.14), and the connection terms from (A.3.17):

$$\partial_\theta \, K^{\phi\theta} \;+\; \frac{1}{2}\,\left(\gamma^{rr}\,\frac{\partial\gamma_{rr}}{\partial x^\theta} \;+\; \gamma^{\theta\theta}\,\frac{\partial\gamma_{\theta\theta}}{\partial x^\theta} \;+\; 3\,\gamma^{\phi\phi}\,\frac{\partial\gamma_{\phi\phi}}{\partial x^\theta}\right)\,K^{\phi\theta}$$

$$= \frac{\alpha M a}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}$$

$$\left(\left(2ra^2\cos^2\theta - 2ra^2\sin^2\theta \;+\; \frac{6ra^4\cos^2\theta\;\sin^2\theta}{\Sigma}\right.\right.$$
$$\left. +\; \cancel{\frac{6ra^4\triangle\cos^2\theta\;\sin^2\theta}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}}\right)$$
$$+\left(\frac{6r^3a^2\cos^2\theta - 4ra^4\cos^2\theta\;\sin^2\theta + 6ra^4\cos^2\theta}{\Sigma}\right.$$
$$\left.\left. -\; \cancel{\frac{6ra^4\triangle\cos^2\theta\sin^2\theta}{\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}}\right)\right)$$

$$= \frac{\alpha M a}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}$$
$$\left(2ra^2\cos^2\theta - 2ra^2\sin^2\theta \;+\; \frac{6r^3a^2\cos^2\theta + 2ra^4\cos^2\theta\;\sin^2\theta + 6ra^4\cos^2\theta}{\Sigma}\right)$$

$$= \frac{2r\alpha M a^3}{\Sigma^2\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}$$
$$\left(\cos^2\theta - \sin^2\theta \;+\; \frac{3r^2\cos^2\theta + a^2\cos^2\theta\;\sin^2\theta + 3a^2\cos^2\theta}{\Sigma}\right)$$

$$= \frac{2r\alpha M a^3\left(4r^2\cos^2\theta - r^2\sin^2\theta + 3a^2\cos^2\theta + a^2\cos^4\theta\right)}{\Sigma^3\left((r^2+a^2)^2 - \triangle a^2\sin^2\theta\right)}, \qquad \text{(A.3.20)}$$

giving the terms for the curvature component $K^{\phi\theta}$, simplified as much as possible.

Combining the two parts (A.3.19) and (A.3.20), gives the full ϕ part of the extrinsic curvature divergence:

$$\begin{aligned} D_b K^{\phi b} &= \partial_r \ K^{\phi r} + \ \partial_\theta \ K^{\phi\theta} \\ &\quad + \ \frac{1}{2} \ \left(\gamma^{rr} \ \frac{\partial \gamma_{rr}}{\partial x^r} \ + \ \gamma^{\theta\theta} \ \frac{\partial \gamma_{\theta\theta}}{\partial x^r} \ + \ 3 \ \gamma^{\phi\phi} \ \frac{\partial \gamma_{\phi\phi}}{\partial x^r} \right) \ K^{\phi r} \\ &\quad + \ \frac{1}{2} \ \left(\gamma^{rr} \ \frac{\partial \gamma_{rr}}{\partial x^\theta} \ + \ \gamma^{\theta\theta} \ \frac{\partial \gamma_{\theta\theta}}{\partial x^\theta} \ + \ 3 \ \gamma^{\phi\phi} \ \frac{\partial \gamma_{\phi\phi}}{\partial x^\theta} \right) \ K^{\phi\theta} \\ &= \frac{2r\alpha M a^3 \left(r^2 - 5r^2 \cos^2\theta - 3a^2 \cos^2\theta - a^2 \cos^4\theta \right)}{\Sigma^3 \left((r^2 + a^2)^2 - \triangle a^2 \sin^2\theta \right)} \\ &\quad - \frac{2r\alpha M a^3 \left(r^2 - 5r^2 \cos^2\theta - 3a^2 \cos^2\theta - a^2 \cos^4\theta \right)}{\Sigma^3 \left((r^2 + a^2)^2 - \triangle a^2 \sin^2\theta \right)} \\ &= 0 \ , \end{aligned} \tag{A.3.21}$$

which shows the momentum constraint (A.3.9) to be completely satisfied.

The momentum constraint is therefore satisfied for the expressions which have been calculated for the curvature components and their derivatives. This implies that the extrinsic curvature terms given by (A.1.32) and (A.1.36), and their derivatives with respect to r and θ, given by equations (A.2.9) and (A.2.14), have been correctly calculated, and can be used confidently for further calculations.

Appendix B

Mathematica Calculations for Brill Mass-Factor

In chapter 5, section 5.2.3, the numerical calculations were carried out using *Mathematica 7.0.* The operations used for these calculations are given here.

B.1 Solving Ordinary Differential Equation

The ordinary differential equation given by equation (5.2.16):

$$\partial_{\rho\rho}\,\psi(\rho) + \frac{1}{\rho}\,\partial_\rho\,\psi(\rho) + \frac{1}{2}A(2\rho^4 - 5\rho^2 + 1)\ e^{-\rho^2}\,\psi(\rho) \ = \ 0\ , \tag{B.1.1}$$

was solved for the conformal factor $\psi(\rho)$, with boundary conditions given by equations (5.2.17) and (5.2.19):

$$\partial_\rho\psi(\rho = 0) \ := 0\ , \tag{B.1.2a}$$

$$\psi(\rho_{max}) \ := 1\ . \tag{B.1.2b}$$

With the factor A specified at the beginning of the calculation, the "NDSolve" operation was used to solve the collection of equations given by (B.1.1) and (B.1.2). A plot was then made of the solution to this collection of equations, along with its multiplicative from inside the integral of M_ρ in equation (5.2.20). The input for the graph given by figure 5.2(a), all contained in a single cell, is given below:

```
   ClearAll[A, Rho, Psi, soln, psi];

   A := 4;

   psi = First[Psi /.
NDSolve[
{Rho Psi"[Rho] + Psi'[Rho]
+ 1/2 A (2 Rho∧4 - 5 Rho∧2 + 1) Rho Exp[-Rho∧2] Psi[Rho] == 0,
Psi'[0.001] == 0, Psi[10] == 1},
Psi, {Rho, 0.001, 10}, MaxSteps -> 20000]];

   Plot[
{psi[Rho], 1/2 A (2 Rho∧4 - 5 Rho∧2 + 1) Rho Exp[-Rho∧2]},
{Rho, 0.001, 4}]

   ClearAll[A];
```

B.2 Graphing Mass-Factor

The mass factor was found, for a range of values of the scalar A, by first solving for the conformal factor $\psi(\rho)$ as in section B.1, and then numerically evaluating the integral for the mass factor M_ρ, given by equation (5.2.20):

$$M_\rho \;=\; \frac{1}{2}A\int_0^4 \rho\,(2\rho^4 - 5\rho^2 + 1)\; e^{-\rho^2}\,\psi(\rho)\,d\rho\;. \tag{B.2.1}$$

A table was created of A versus the solution to (B.2.1), for a range of values of A, which was then plotted using the "`ListPlot`" operation. Again, contained in one cell, the input is given below, for the graph in figure 5.3(a):

```
   ClearAll[A, Rho, Psi, soln, psi, g]



   g = Table[
{A,

   psi = First[Psi /.
NDSolve[
{Rho Psi"[Rho] + Psi'[Rho]
+ 1/2 A (2 Rho∧4 - 5 Rho∧2 + 1) Rho Exp[-Rho∧2] Psi[Rho] == 0,
Psi'[0.001] == 0, Psi[10] == 1},
Psi, {Rho, 0.001, 10}, MaxSteps -> 20000]];

   NIntegrate[
1/2 A (2 Rho∧4 - 5 Rho∧2 + 1) Rho Exp[-Rho∧2] psi[Rho],
{Rho, 0.001, 4}]},

   {A, 0, 8, 0.1}];



   ListPlot[g,
PlotRange -> {{0, 8}, {-10, 10}},
AxesLabel -> {Style["A"], Style["M-Rho (A)"]}]
```

List of Symbols

$: \rightarrow$	Definition of an operation, sending elements of one set to those of another.
$:=$	Defined to be equal to.
$\equiv$	Equivalent to.
$\simeq$	Approximately equal to.
$\in$	"is an element of".
$\subset$	"is a subset/subspace of".
$\forall$	"for all of".
$\Leftrightarrow$	Forward and backward implication - "if and only if" - "it is necessary and sufficient that".
$\Rightarrow$	Forward implication - "only if" - "it is necessary that".
$\Leftarrow$	Backward implication - "if" - "it is sufficient that".
$\mid \ \mid$	Absolute value - the positive root of the square of a real number - the positive root of the inner product of a vector/covector with itself.
$\langle \ , \rangle$	Inner product of two elements of a vector space, see equation (1.2.15), page 9
$\{ \ \}$	Set of elements.
$\{i \mid h(i)\}$	The set of all elements i which satisfy the condition $h(i)$, see e.g. equation (1.2.76), page 22
$[\ ,]^a$	Commutator bracket of two vector fields, see equation (1.2.46), page 15
a	Kerr rotation factor, see equation (1.3.34), page 37
a	Arbitrary constant in Bowen-York Linear curvature, see equation (2.2.60), page 61
$a, b, ...$	Spatial indices for the $3+1$ time slices, to be summed over 3 spatial coordinates, page 42

$A(x,y)$	Metric component for *Baker & Puzio* [5], see equation (3.1.1), page 74
A	Scalar constant used in the Brill axially-symmetric metric (3.3.1), page 96
A^{ab}	Trace-free part of the extrinsic curvature, see equation (2.2.10), page 53
$\bar{A}^{ab}$	Conformal trace-free part of the extrinsic curvature, see equation (2.2.20), page 55
A_L^{ab}	Longitudinal, trace-free part of extrinsic curvature, see equation (2.2.11), page 53
A_{TT}^{ab}	Transverse, trace-free part of the extrinsic curvature, see equation (2.2.11), page 53
$\bar{A}_{TT}^{ab}$	Conformal transverse trace-free part of the extrinsic curvature, see equation (2.2.24), page 56
α	Lapse of coordinate time in the $3+1$ formalism, see equation (2.1.5), page 41
$\bar{\alpha}$	Densitised lapse, see equation (2.2.45), page 59
$\alpha, \beta, ...$	Space-time indices, to be summed over 4 coordinates, page 26
$B(x,y)$	Metric component for *Baker & Puzio* [5], see equation (3.1.1), page 74
β^a	Shift vector, shift of spatial coordinates between the $3+1$ time slices, see equation (2.1.11), page 43
c	Speed of light, set equal to one for geometrized units, page 25
ds	Line element - infinitesimal distance on a differential manifold, see equation (1.2.21), page 10
D_v	Directional derivative with respect to the vector field v^a, for section 1.2, see equation (1.2.25), page 11
D_a	Spatial covariant derivative on a $3+1$ time slice, see equation (2.1.20), page 45
$\bar{D}_a$	Conformal, spatial covariant derivative, see equation (2.2.4), page 52
D_F^2	Flat space Laplacian, contraction of two flat space covariant derivatives, see equation (5.1.9), page 131
∂_a	Gradient operator, or the set of partial derivatives, with respect to the coordinate vectors x^a: $\partial_a \equiv \frac{\partial}{\partial x^a}$, see equation (1.2.26), page 11
∂_t	Time derivative for the $3+1$ evolution equations, see equation (2.1.19), page 44

∇_w	Covariant derivative with respect to the vector field w^a, page 12
∇_a	Covariant derivative operator, page 12
δ^i_j	Kronecker delta, takes the value 1 for $i = j$ and 0 for $i \neq j$, see equation (1.2.12), page 8
$\triangle$	Simplification term for the Kerr metric in Boyer-Lindquist coordinates, see equation (1.3.35), page 37
$\triangle_L$	Vector Laplacian, related to conformal Killing form, see equation (2.2.16), page 54
$\bar{\triangle}_L$	Conformal vector Laplacian, related to conformal Killing form, see equation (2.2.25), page 56
$\vec{e}_i$	Basis vector or basis vector fields, see equation (1.2.9), page 7
ϵ_{abc}	Levi-Civita antisymmetric tensor, see equation (2.2.57), page 61
$\eta_{\mu\nu}$	Minkowski "flat" space-time metric, see equation (1.3.6), page 26
η^a	Killing vector in *Dain* [19], page 76
η	Norm of the Killing vector η^a in *Dain* [19], page 76
$\mathbb{E}^n$	Euclidean "flat" space, of dimension n., page 4
g_{ab}	Metric on a differential manifold, see equation (1.2.15), page 9
g^{ab}	Inverse metric on a differential manifold, see equation (1.2.19), page 9
$g_{\mu\nu}$	Space-time metric, page 31
g	The determinant of the space-time metric, see equation (1.2.59), page 18
G	Gravitational constant, set equal to one for geometrized units, page 28
$G_{\mu\nu}$	The Einstein-Hilbert tensor, see equation (1.3.17), page 31
γ	Differential curve on a manifold for section 1.2, see equation (1.2.5), page 6
γ	Lorentz factor for section 1.3, see equation (1.3.2), page 25
γ	The determinant of the spatial metric, page 52
γ_{ab}	Spatial metric of the $3+1$ time slices, see equation (2.1.7), page 42
γ^{ab}	Inverse spatial metric of the $3+1$ time slices, see equation (2.1.8), page 42
γ^μ_a	Projection operator onto the $3+1$ time slices, see equation (2.1.9), page 42
$\bar{\gamma}$	The determinant of the conformal metric, page 52

$\bar{\gamma}_{ab}$	Conformal spatial metric, see equation (2.2.1), page 51
$\tilde{\gamma}_{ab}$	Natural conformal metric, i.e. with determinant of 1, see equation (2.2.3), page 52
Γ^c_{ab}	Connection coefficients - Christoffel Symbols when associated with a metric, see equation (1.2.36), page 13
$\bar{\Gamma}^c_{ab}$	Conformal connection coefficients, see equation (2.2.4), page 52
${}^{(4)}\Gamma^\alpha$	Gauge source function for harmonic coordinates, see equation (2.3.13), page 65
$\bar{\Gamma}^a$	Conformal connection functions, see equation (2.3.28), page 67
$\hat{H}_{ab}$	Conformal extrinsic curvature in *Brandt & Seidel* [12], see equation (3.1.8), page 75
j^a	Momentum density for the $3+1$ formalism, see equation (2.1.29), page 46
J	Jacobian matrix for section 1.2, see equation (1.2.60), page 18
J	Angular momentum, page 37
K_{ab}	Extrinsic curvature of a space-like hypersurface, see equation (2.1.16), page 44
$\bar{K}_{ab}$	Conformal extrinsic curvature, note from equation (2.2.18), this usually only makes sense for $K = 0$, see e.g. equation (2.2.59), page 61
K_L^{ab}	Longitudinal extrinsic curvature, see equation (2.2.8), page 53
K_T^{ab}	Transverse (i.e. divergence-free) extrinsic curvature, see equation (2.2.8), page 53
K	Mean curvature, trace of extrinsic curvature, see equation (2.1.21), page 45
k^{ab}	The difference between the Kerr and Bowen-York extrinsic curvature tensors, see equation (4.3.1), page 123
$\bar{K}_{BY}^{ab}$	The angular momentum part of the confromal Bowen-York extrinsic curvature, see equation (4.3.1), page 123
K_K^{ab}	The extrinsic curvature of the constant Boyer-Lindquist time-slice of the Kerr metric, see equation (4.3.1), page 123
$(KW)^{ab}$	Killing form, with respect to the vector field W^a, see equation (2.2.9), page 53
$(LW)^{ab}$	Conformal Killing form, with respect to the vector field W^a, see equation (2.2.12), page 54

$\left(\bar{L}W\right)^{ab}$	Conformally transformed conformal Killing form, see equation (2.2.24), page 56
$\mathcal{L}_v$	The Lie derivative with respect to the vector field v^a, see equation (1.2.43), page 14
ln	The natural logarithm.
M	Differential manifold for section 1.2, page 5
M	Schwarzschild mass, page 33
M_ρ	"Mass Factor" associated with simplified Brill space in section 5.2, see equation (5.2.10), page 135
M^{ab}	Symmetric trace-free tensor, see equation (2.2.32), page 57
$\bar{M}^{ab}$	Conformal symmetric trace-free tensor, see equation (2.2.36c), page 57
$\mu, \nu, ...$	Space-time indices, to be summed over 4 coordinates, page 26
n^μ	Unit time-like normal to the $3+1$ time slices, see equation (2.1.4), page 41
ω	Scalar potential for a transverse trace-free tensor in *Dain* [19], see equation (3.1.14), page 76
Ω_μ	Future-pointing time-like covector field, used for the $3+1$ foliations, see equation (2.1.1), page 41
$\vec{\omega}^i$	Basis covector field, page 7
$O(\ )$	Terms of order ..., see e.g. equation (1.3.32), page 36
P^a	Linear momentum, page 60
Φ	Newtonian gravitational potential for section 1.3.2, see equation (1.3.10), page 29
Φ	Differential function related to scalar curvature, from *Brill* [13], for chapter 5, see equation (5.1.8), page 131
ϕ	Azimuthal angle, in a spherical or cylindrical-polar type coordinate system, pages 10, 87
ψ	Conformal factor, see equation (2.2.1), page 51
q^a	The space-like unit normal vector to a sphere of constant radius, page 60
$q(\rho, z)$	Differential function, satisfying specific conditions (3.3.2), used in the Brill axially-symmetric metric (3.3.1), page 96
$\mathbb{R}$	Set of Real numbers, page 4

r	Outward directed radial coordinate, in a spherical-polar type coordinate system, page 10
R	Scalar curvature, see equation (1.2.72), page 20
$\bar{R}$	Conformal scalar curvature, see equation (2.2.6), page 52
R_{ab}	Ricci curvature tensor, see equation (1.2.71), page 20
$\bar{R}_{ab}$	Conformal Ricci curvature tensor, see equation (2.2.5), page 52
$R^a{}_{bcd}$	Riemann curvature tensor, see equation (1.2.66), page 19
$\bar{R}^a{}_{bcd}$	Conformal Riemann curvature tensor, see equation (2.2.5), page 52
ρ	Energy density for the $3+1$ formalism, see equation (2.1.26), page 46
ρ	Newtonian mass density for section 1.3.2, see equation (1.3.11), page 29
ρ	Radial coordinate from central axis, in a cylindrical-polar type coordinate system, page 87
S_{ab}	Spatial part of stress-energy tensor, decomposed into the $3+1$ formalism, see equation (2.1.38), page 48
S	Trace of spatial part of stress-energy tensor, see equation (2.1.38), page 48
S^a	Solution for a transverse trace-free extrinsic curvature in *Dain* [19], page 76
Σ_t	Time slices of the $3+1$ foliation, given by space-like hypersurfaces, page 41
Σ	Simplification term for the Kerr metric in Boyer-Lindquist coordinates, see equation (1.3.35), page 37
span$\{\}$	Span of tangent vector fields - set of all linear combinations of given tangent vectors at each point, see equation (1.2.76), page 22
$s.t.$	"such that".
$T_{\mu\nu}$	Stress-energy tensor, page 28
T	Trace of the stress-energy tensor, page 31
$\mathcal{T}_b^a$	Tensor density, page 18
t	Time parameter for the $3+1$ foliation, see equation (2.1.1), page 41
t	Time coordinate for the $3+1$ formalism, see equation (2.1.12), page 43
t^μ	Time vector in the $3+1$ formalism, see equation (2.1.11), page 43
θ	Inclination angle, measured from zenith, in a spherical-polar type coordinate system, page 10

u — "Potential function" for a transverse trace-free tensor in *Baker & Puzio* [5], see equation (3.1.5), page 74

u_{ab} — Trace-free part of the spatial metric time derivatie, see equation (2.2.39), page 58

$\bar{u}_{ab}$ — Conformal time derivative of the spatial metric, see equation (2.2.41), page 58

V — Scalar potential for a transverse trace-free tensor, in spherical-polar coordinates, in flat 3-dimensional space, see equation (3.2.11), page 85

$\vec{v}$ — Tangent vector or vector field, see equation (1.2.9), page 7

v^a — Vector field - abstract index notation, page 7

W — Scalar potential for a transverse trace-free tensor, in spherical-polar coordinates, in flat 3-dimensional space, see equation (3.2.13), page 85

w — Scalar potential for a transverse trace-free tensor, with respect to a Brill type metric (3.3.1), see equation (3.3.10), page 98

X — Scalar potential for a transverse trace-free tensor, in cylindrical-polar coordinates, in flat 3-dimensional space, see equation (3.2.23), page 88

x — Cartesian type coordinate, page 35

Y — Scalar potential for a transverse trace-free tensor, in cylindrical-polar coordinates, in flat 3-dimensional space, see equation (3.2.25), page 88

y — Cartesian type coordinate, page 35

z — Cartesian type coordinate, or axial coordinate in a cylindrical-polar type coordinate system, pages 35, 87

$\vec{\zeta}$ — Covector field, see equation (1.2.10), page 7

ζ_a — Covector field - abstract index notation, page 8

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